

IRREDUCIBILITY OF MODULI SPACES OF VECTOR BUNDLES ON K3 SURFACES

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ABSTRACT. In this paper, we show the moduli spaces of stable sheaves on K3 surfaces are irreducible symplectic manifolds, if the associated Mukai vectors are primitive. More precisely, we show that they are related to the Hilbert scheme of points. We also compute the period of these spaces. As an application of our result, we discuss Montonen-Olive duality in Physics. In particular our computations of Euler characteristics of moduli spaces are compatible with Physical computations by Minahan et al.

0. INTRODUCTION

0.1. Main result. Let X be a projective K3 surface defined over \mathbb{C} and H an ample divisor on X . Let ω be the fundamental class of X . Let E be a coherent sheaf on X . By the identification $H^4(X, \mathbb{Z}) \cong \mathbb{Z}\omega$, we regard the second Chern class $c_2(E)$ as an integer. Since $(c_1(E)^2)$ is even, the second Chern character $\text{ch}_2(E)$ belongs to \mathbb{Z} . We define the Mukai vector of E by

$$\begin{aligned} v(E) &:= \text{ch}(E)\sqrt{\text{td}_X} \\ &= \text{rk}(E) + c_1(E) + (\text{rk}(E) + \text{ch}_2(E))\omega \in H^*(X, \mathbb{Z}), \end{aligned} \quad (0.1)$$

where we identify $H^0(X, \mathbb{Z})$ with \mathbb{Z} and $\text{td}_X = 1 + 2\omega$ is the Todd class of X . For an element $v \in H^*(X, \mathbb{Z})$, we denote the 0-th component $v_0 \in H^0(X, \mathbb{Z})$ by $\text{rk } v$ and the second component $v_1 \in H^2(X, \mathbb{Z})$ by $c_1(v)$. We set $\ell(v) := \gcd(\text{rk } v, c_1(v)) \in \mathbb{Z}_{\geq 0}$. Then v is written as $v = \ell(v)(r + \xi) + a\omega$, where $r \in \mathbb{Z}$, $\xi \in H^2(X, \mathbb{Z})$ and $r + \xi$ is primitive. We denote the moduli space of stable sheaves E of $v(E) = v$ by $M_H(v)$. If v is primitive and H is general in the ample cone $\text{Amp}(X)$ of X (i.e. there are at most countable number of hyperplanes $W_n \subset H^2(X, \mathbb{Q})$, $n = 1, 2, \dots$ which depends on v and H belongs to $\text{Amp}(X) \setminus \cup_n W_n$ [Y3]), then $M_H(v)$ is a smooth projective scheme. In [Mu1], Mukai showed that $M_H(v)$ has a symplectic structure. In order to get more precise information, Mukai [Mu2] introduced a quite useful notion called Mukai lattice $(H^*(X, \mathbb{Z}), \langle \ , \ \rangle)$, where the pairing is defined by

$$\begin{aligned} \langle x, y \rangle &:= - \int_X x^\vee y \\ &= \int_X (x_1 y_1 - x_0 y_2 - x_2 y_0), \end{aligned} \quad (0.2)$$

where $x_i \in H^{2i}(X, \mathbb{Z})$ (resp. $y_i \in H^{2i}(X, \mathbb{Z})$) is the $2i$ -th component of x (resp. y) and $x^\vee = x_0 - x_1 + x_2$. Hence $\langle \ , \ \rangle$ is an integral primitive bilinear form on $H^*(X, \mathbb{Z})$. By the language of this lattice, we can write down Riemann-Roch theorem in a simple form: We set

$$\chi(E, F) := \sum_{i=0}^2 \dim \text{Ext}^i(E, F) \quad (0.3)$$

for coherent sheaves E and F . Then Riemann-Roch theorem implies that

$$\chi(E, F) = -\langle v(E), v(F) \rangle. \quad (0.4)$$

In particular we get that $\dim M_H(v) = \langle v^2 \rangle + 2$.

If v is a primitive isotropic vector, then $M_H(v)$ is a surface with a symplectic structure. Mukai proved that $M_H(v)$ is a K3 surface and described the period in terms of Mukai lattice. If v is a primitive Mukai vector of $\langle v^2 \rangle > 0$, then $M_H(v)$ is a higher dimensional symplectic manifold. If $\text{rk } v = 1$, then $M_H(v)$ is the Hilbert scheme of points on X . Indeed every torsion free sheaf of rank 1 is give by $I_Z \otimes L$, where I_Z is the ideal sheaf of a 0-dimensional subscheme of X and L is a line bundle of $c_1(L) = c_1(v)$. Beauville [B] proved that it is an example of higher dimensional irreducible symplectic manifold. For an irreducible symplectic manifold, Beauville [B] defined the period and proved local Torelli theorem. As an example, he also computed the period of Hilbert scheme of points on X . For higher rank cases, Mukai [Mu3] (rank 2 case), O'Grady [O1] ($\ell(v) = 1$ case) and the author [Y5] ($\langle v^2 \rangle > 2\ell(v)^2$ or $\ell(v) = 1$ case) proved that $M_H(v)$ is an irreducible symplectic manifold and described the period of $M_H(v)$ in terms of Mukai lattice. For classification of $M_H(v)$, it is important to determine the period. Indeed, it is a birational invariant

([Mu3]), and affirmative solution of Torelli conjecture will imply that an irreducible symplectic manifold is determined by its period, up to birational equivalence.

In this paper, by using [Y5] extensively, we prove the following theorem, which is expected by many people (for example, see [D], [Mu3], [O1]).

Theorem 0.1. *Let v be a primitive Mukai vector such that $\text{rk } v > 0$ and $c_1(v) \in \text{NS}(X)$.*

- (1) *$M_H(v)$ is not empty for a general ample divisor H if and only if $\langle v^2 \rangle \geq -2$.*
- (2) *Assume that $\langle v^2 \rangle \geq -2$. Then for a general ample divisor H ,*
 - (2-1) *$M_H(v)$ is obtained by compositions of deformations and birational transformations from $\text{Hilb}_X^{\langle v^2 \rangle/2+1}$. In particular $M_H(v)$ is an irreducible symplectic manifold.*
 - (2-2) *Let $B_{M_H(v)}$ be Beauville's bilinear form on $H^2(M_H(v), \mathbb{Z})$. Then*

$$\theta_v : (v^\perp, \langle \cdot, \cdot \rangle) \rightarrow (H^2(M_H(v), \mathbb{Z}), B_{M_H(v)})$$

is an isometry which preserves Hodge structures for $\langle v^2 \rangle \geq 2$, where $\theta_v : v^\perp \rightarrow H^2(M_H(v), \mathbb{Z})$ is the canonical homomorphism defined by using a quasi-universal family.

Here we only use deformations of $M_H(v)$ induced by deformation of complex structures of X .

Since birationally equivalent Calabi-Yau manifolds have the same Hodge numbers ([Ba],[De-L]), we get the following Corollary.

Corollary 0.2. *Keep the notations as above. Then $h^{p,q}(M_H(v)) = h^{p,q}(\text{Hilb}_X^{\langle v^2 \rangle/2+1})$. In particular, $\chi(M_H(v)) = \chi(\text{Hilb}_X^{\langle v^2 \rangle/2+1})$.*

In [V-W], Vafa and Witten considered a partition function $Z_r^\alpha(\tau)$, $\alpha \in H^2(X, \mathbb{Z})$, $\tau \in \mathbb{H} := \{z \in \mathbb{C} | \Im z > 0\}$ associated with $N = 4$ super symmetric Yang-Mills theory on a 4 manifold X . Under suitable vanishing conditions (e.g. $H^0(X, \text{ad}(E) \otimes K_X) = 0$), it is related to ‘‘Euler characteristics’’ of moduli spaces of vector bundles. For a K3 surface case, $Z_r^\alpha(\tau)$ is given by

$$Z_r^\alpha(\tau) = \sum_{\substack{\text{rk } v=r \\ c_1(v)=\alpha}} ‘‘\chi(M_H(v))’’ q^{\langle v^2 \rangle/2r}, \quad (0.5)$$

where ‘‘ $\chi(M_H(v))$ ’’ is a kind of ‘‘Euler characteristics’’ of a suitable compactification of $M_H(v)$. Recently, this invariant was computed in [MNVW]. In section 4, by using Corollary 0.2, we shall check that their computation coincides with the Euler characteristics of $M_H(v)$, if v is primitive.

For a non-primitive Mukai vector, we have the following existence condition.

Corollary 0.3. *Let v be a Mukai vector of $\text{rk } v > 0$. Then there is a semi-stable sheaf E of $v(E) = v$ with respect to a general ample divisor H if and only if $v = nw$, $n \in \mathbb{Z}$, $w \in H^*(X, \mathbb{Z})$ with $\langle w^2 \rangle \geq -2$.*

0.2. Outline of the proof. We shall explain how to prove (2-1) of this theorem. In [Y3], we discussed chamber structure of polarizations. Let $v = l(r + \xi) + a\omega$, $\xi \in \text{NS}(X)$ be a Mukai vector of $l = \ell(v)$ and $r > 0$. We choose an ample divisor H on X which does not lie on walls with respect to v . Then

- (†) for every μ -semi-stable sheaf E of $v(E) = v$, if $F \subset E$ satisfies $(c_1(F), H)/\text{rk } F = (c_1(E), H)/\text{rk } E$, then $c_1(F)/\text{rk } F = c_1(E)/\text{rk } E$.

Thus $v(F) = l'(r + \xi) + a'\omega$ for some l', a' . In particular, if v is primitive, then $M_H(v)$ is compact. Let $\mathcal{M}(v)$ be the stack of coherent sheaves E of $v(E) = v$. We shall fix a general ample divisor H with respect to v . $\mathcal{M}(v)^{\mu ss}$ (resp. $\mathcal{M}(v)^{\mu s}$) denotes the open substack of $\mathcal{M}(v)$ consisting of μ -semi-stable sheaves (resp. μ -stable sheaves).

In [Y5], we proved Theorem 0.1 under the assumption $\langle v^2 \rangle > 2l^2$ or $l = 1$. Hence we may assume that $\langle v^2 \rangle \leq 2l^2$ and $l > 1$. However for convenience sake of the reader, we only use the results for $l = 1$ case. We note that isometry group $\text{O}(H^*(X, \mathbb{Z}))$ of Mukai lattice acts transitively on the set $V_n := \{x \in H^*(X, \mathbb{Z}) | x \text{ is primitive}, \langle x^2 \rangle = 2n\}$ and $\text{O}(H^*(X, \mathbb{Z}))/\pm 1$ is generated by the following 3 kinds of isometries:

1. Translation: For $N \in \text{Pic}(X)$,

$$\begin{aligned} T_N : H^*(X, \mathbb{Z}) &\rightarrow H^*(X, \mathbb{Z}) \\ x &\mapsto \text{ch}(N)x \end{aligned} \quad (0.6)$$

is an isometry.

2. $\text{O}(H^2(X, \mathbb{Z}))$ acts on $\text{O}(H^*(X, \mathbb{Z}))$.
3. Reflection: For a (-2) -vector $v_1 \in H^*(X, \mathbb{Z})$,

$$\begin{aligned} R_{v_1} : H^*(X, \mathbb{Z}) &\rightarrow H^*(X, \mathbb{Z}) \\ x &\mapsto x + \langle x, v_1 \rangle v_1 \end{aligned} \quad (0.7)$$

is an isometry.

Therefore it is very important to understand reflections.

Geometric realization of reflections: As we shall see in Corollary 3.3, a reflection is realized as a Fourier-Mukai transform. Here we shall explain a special case. Let E_1 be a stable vector bundle of $\text{Ext}^1(E_1, E_1) = 0$ (E_1 is called exceptional vector bundle). Since $\text{Ext}^2(E_1, E_1) \cong \text{Hom}(E_1, E_1)^\vee = \mathbb{C}$, Riemann-Roch theorem implies that $\langle v(E_1), v(E_1) \rangle = -\chi(E_1, E_1) = -2$. Thus $v_1 := v(E_1)$ is a (-2) -vector. Let E be a stable vector bundle of $v(E) = v$. Assume that

- (a) $\text{Ext}^i(E_1, E) = 0$, $i = 1, 2$.
- (b) $\phi : E_1 \otimes \text{Hom}(E_1, E) \rightarrow E$ is surjective and $\ker \phi$ is stable.

Then $w := v(\ker \phi)$ is given by $\chi(E_1, E)v_1 - v = -(v + \langle v, v_1 \rangle v_1)$. Thus $-v(\ker \phi)$ is the (-2) -reflection of v by v_1 . Hence under conditions (a) and (b), (-2) -reflection of Mukai lattice induces a birational map $M_H(v) \cdots \rightarrow M_H(w)$. Replacing $\ker \phi$ by $\text{coker}(\phi^\vee : E^\vee \rightarrow E_1^\vee \otimes \text{Hom}(E_1, E)^\vee)$, we may replace (b) by the condition (b'):

- (b') $\phi : E_1 \otimes \text{Hom}(E_1, E) \rightarrow E$ is surjective in codimension 1 and $\ker \phi$ is stable.

Thus under (a) and (b'), we have a birational map $M_H(v) \cdots \rightarrow M_H(w^\vee)$. We would like to apply this story for a suitable pair of v_1 and v which satisfy $\langle v_1, v \rangle = -1$. In order to get condition (b'), we shall prove the following key lemma which was proved under the assumption $lr < r_1 < (l+1)r$ in [Y5, Prop. 4.5].

Lemma 0.4. *Let (X, H) be a polarized smooth projective surface of $\text{NS}(X) = \mathbb{Z}H$. Let (r_1, d_1) and (r, d) be pairs of integers such that $r_1, r > 0$ and $dr_1 - rd_1 = 1$. We assume that $lr < r_1$. Let E_1 be a μ -stable vector bundle of $\text{rk}(E_1) = r_1$ and $\deg(E_1) = d_1$, where $\deg(E_1) = (c_1(E_1), H)/(H^2)$.*

- (1) *Let E be a μ -stable sheaf of $\text{rk}(E) = lr$ and $\deg(E) = ld$. Then every non-zero homomorphism $\varphi : E_1 \rightarrow E$ is surjective in codimension 1 and $\ker \varphi$ is a μ -stable sheaf.*
- (2) *Let E' be a μ -stable vector bundle of $\text{rk}(E') = r_1 - lr$ and $\deg(E') = d_1 - ld$. Let $\phi : E' \rightarrow E_1$ be a non-zero homomorphism. Then ϕ is injective and $E := \text{coker } \phi$ is a μ -semi-stable sheaf.*

For the proof of this lemma, we use the following fact:

- We consider the triangle in \mathbb{R}^2 with vertices $(0, 0)$, $(r_1 - lr, d_1 - ld)$ and (r_1, d_1) . Then there is no integral point in the interior of this triangle.

Indeed, this condition gives a strong restriction on homomorphisms φ , ϕ and the Harder-Narasimhan polygons of $\ker \varphi$ and $\text{coker } \phi$. The proof will be done in Preliminaries.

In order to use this lemma, we need to compare $\mathcal{M}(v)^{\mu ss}$ and $\mathcal{M}(v)^{\mu s}$. More precisely, we need dimension counting of various constructible substacks of $\mathcal{M}(v)^{\mu ss}$. Technically this is the most important part in this paper. In [D-L], Drezet and Le Potier computed the dimension of the substack of non-semi-stable sheaves. In their computation, the existence of exceptional vector bundle is very important. In our case, we concentrate our consideration on $\mathcal{M}(v)^{\mu ss}$. By our assumption on H , exceptional vector bundle E of $v(E) = r + \xi + b\omega$, $b \in \mathbb{Z}$ is important. Hence we divide our proof into two cases:

- A. There is no (-2) vector of the form $r + \xi + b\omega$, i.e. $((\xi^2) + 2)/2r \notin \mathbb{Z}$.
- B. There is a (-2) vector of the form $r + \xi + b\omega$, i.e. $((\xi^2) + 2)/2r \in \mathbb{Z}$.

In section 2, we treat case A. In particular, we prove the following inequality:

$$\dim(\mathcal{M}(v)^{\mu ss} \setminus \mathcal{M}(v)^{\mu s}) \leq \langle v^2 \rangle. \quad (0.8)$$

Then we can apply Lemma 0.4. For suitable choice of

- 1. a primitive Mukai vector $v := l(r + dH) + a\omega$ on (X, H) and
- 2. an exceptional vector bundle E_1 of $v(E_1) = r_1 + d_1H + a_1\omega$,

we can construct a birational map $M_H(v) \cdots \rightarrow M_H(w^\vee)$ sending a general μ -stable vector bundle $E \in M_H(v)$ to $F := \text{coker}(E^\vee \rightarrow \text{Hom}(E_1, E)^\vee \otimes E_1^\vee) \in M_H(w^\vee)$ where H and v' satisfy that (1) $(H^2)/2 = (r_1 \langle v^2 \rangle / 2l + r)r_1 / l - r^2 > 0$, (2) $\langle v_1, v \rangle = -1$ and (3) $\ell(w^\vee) = 1$ and hence Theorem 0.1 holds for $M_H(w^\vee)$. We remark that we need to choose a sufficiently large r_1 for the condition (1). In the same way as in [Y5, 4.3], we get Theorem 0.1 for case A. More precisely, considering deformations of $M_H(v)$ induced by deformations of (X, H) and translations T_N , we can reduce the problem to this situation.

In section 3, we treat case B. If $\langle v^2 \rangle \geq 2l^2$, then we also have the inequality (0.8), and hence the same proof as in case A works. If $\langle v^2 \rangle < 2l^2$, then it is known that there is no μ -stable sheaf. Hence we can not apply Lemma 0.4 in this form. When the (-2) vector is $v(\mathcal{O}_X) = 1 + \omega$, T. Nakashima found the following fact:

We set $v = l - a\omega$. Then the inequality $0 \leq \langle v^2 \rangle < 2l^2$ implies that $0 \leq a < l$. We assume that $a \geq 2$. Let E be a μ -stable vector bundle of $v(E) = a - l\omega$. Then $H^0(X, E) = H^2(X, E) = 0$ and $\dim H^1(X, E) = l - a$. We consider the universal extension (another example of reflection)

$$0 \rightarrow E \rightarrow E' \rightarrow H^1(X, E) \otimes \mathcal{O}_X \rightarrow 0. \quad (0.9)$$

It is easy to see that E' is a stable vector bundle and we get an immersion $M_H(a - l\omega)^{\mu s, loc} \hookrightarrow M_H(v)$, where $M_H(a - l\omega)^{\mu s, loc}$ is the open subscheme of $M_H(a - l\omega)$ consisting of μ -stable vector bundles.

This result can be easily extended to general cases. Hence what we should do is to prove the irreducibility of $M_H(v)$ and the classification of $M_H(v)$ consisting of non-locally free sheaves. By similar dimension counting as in case A, we shall classify non-locally free components and prove the irreducibility of $M_H(v)$. The classification of non-locally free components of $M_H(v)$ is described as follows:

Proposition 0.5. *Keep the notations in Theorem 0.1. Then $M_H(v)$ consists of non-locally free sheaves if and only if $\text{rk } v = 1$, $v = (\text{rk } v_0)v_0 - \omega$ or $v = l - \omega$, where v_0 is a Mukai vector of $\langle v_0^2 \rangle = -2$. For these spaces, $M_H(v) \cong \text{Hilb}_X^{\langle v^2 \rangle/2+1}$.*

1. PRELIMINARIES

1.1. Notation. Except sections 1.5.1 and 1.5.2, we assume that X is a K3 surface. For a scheme S , $p_S : S \times X \rightarrow S$ denotes the projection. For a Mukai vector v , we fix a general ample divisor H which satisfies (†) in section 0.2. Obviously for any subsheaf $E' \subset E$ of μ -semi-stable sheaf E of $v(E) = v$, if $c_1(E')/\text{rk } E' = c_1(E)/\text{rk } E$, then $v(E')$ also satisfies (†). $\mathcal{M}(v)$, $\mathcal{M}(v)^{\mu ss}$ and $\mathcal{M}(v)^{\mu s}$ are stacks in section 0.2. $\mathcal{M}(v)^{ss}$ and $\mathcal{M}(v)^s$ denote the open substack of $\mathcal{M}(v)$ consisting of semi-stable sheaves and stable sheaves respectively. Since $\mathcal{M}(v)^{\mu ss}$ is bounded, it is a quotient stack of an open subscheme of some quotient scheme by some general linear group (see Appendix). Hence our dimension counting of substack of $\mathcal{M}(v)^{\mu ss}$ can be regarded as that of subscheme of some quotient scheme. $M_H(v)^{\mu s}$ (resp. $M_H(v)^{loc}$) be the open subscheme of $M_H(v)$ consisting of μ -stable sheaves (resp. stable vector bundles).

Mukai homomorphism: Let \mathcal{E} be a quasi-universal family of similitude ρ on $M_H(v) \times X$, that is, $\mathcal{E}_{\{E\} \times X} \cong E^{\oplus \rho}$ for all $E \in M_H(v)$ ([Mu3]). By using \mathcal{E} , Mukai constructed a natural homomorphism

$$\theta_v : v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z})_f$$

defined by

$$\theta_v(x) := \frac{1}{\rho} \left[p_{M_H(v)*}((\text{ch } \mathcal{E}) \sqrt{\text{td}_X} x^\vee) \right]_1,$$

where $H^2(M_H(v), \mathbb{Z})_f$ is the torsion free quotient of $H^2(M_H(v), \mathbb{Z})$. We note that θ_v does not depend on the choice of a quasi-universal family.

1.2. Some results from [Y5]. We collect some results which are necessary to prove Theorem 0.1.

Theorem 1.1. *Let $v = l(r + \xi) + a\omega$, $\xi \in H^2(X, \mathbb{Z})$ be a primitive Mukai vector such that $l = \ell(v)$ and $r > 0$. If $\langle v^2 \rangle/2 \geq l^2$ or $l = 1$, and H is general, then $M_H(v)$ is obtained by compositions of deformations and birational transformations from $\text{Hilb}_X^{\langle v^2 \rangle/2+1}$. In particular, $M_H(v)$ is an irreducible symplectic manifold. Let $B_{M_H(v)}$ be Beauville's bilinear form on $H^2(M_H(v), \mathbb{Z})$. If $\langle v^2 \rangle/2 > l^2$, or $l = 1$ and $\langle v^2 \rangle/2 = 1$, then*

$$\theta_v : (v^\perp, \langle \ , \ \rangle) \rightarrow (H^2(M_H(v), \mathbb{Z}), B_{M_H(v)})$$

is an isometry which preserves Hodge structures for $\langle v^2 \rangle \geq 2$.

When $l = \ell(v) = 1$, this theorem was first proved by O'Grady [O1]. In this paper, we only use this theorem for the case where $\ell(v) = 1$.

The following is essentially due to O'Grady [O1]. We can see a different proof based on Göttsche and Huybrechts' argument [G-H] in [Y7].

Proposition 1.2 ([Y5, Prop. 1.1]). *Let X_1 and X_2 be K3 surfaces, and let $v_1 := l(r + \xi_1) + a_1\omega \in H^*(X_1, \mathbb{Z})$ and $v_2 := l(r + \xi_2) + a_2\omega \in H^*(X_2, \mathbb{Z})$ be primitive Mukai vectors such that (1) $r, l > 0$, (2) $r + \xi_1$ and $r + \xi_2$ are primitive, (3) $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 2s$, and (4) $a_1 \equiv a_2 \pmod{l}$. Then $M_{H_1}(v_1)$ and $M_{H_2}(v_2)$ are deformation equivalent. In particular, $M_{H_1}(v_1)$ is an irreducible symplectic manifold and θ_{v_1} is an isometry of Hodge structures if and only if $M_{H_2}(v_2)$ and θ_{v_2} have the same properties.*

Lemma 1.3 ([Y5, Lem. 5.1]). *Let $x_1, x_2, x_3, y_1, y_2, y_3$ be integers such that $x_1, x_2, x_3 > 0$ and $y_1x_3 - x_1y_3 = 1$. If*

$$\frac{y_1}{x_1} > \frac{y_2}{x_2} > \frac{y_3}{x_3}, \tag{1.1}$$

then $x_2 \geq x_1 + x_3$.

Lemma 1.4 ([Y5, Lem. 4.1]). *Let (X, H) be a polarized smooth projective surface of $\text{NS}(X) = \mathbb{Z}H$. Let (r_1, d_1) and (r, d) be pairs of integers such that $r_1, r > 0$ and $dr_1 - rd_1 = 1$. Let E_1 be a μ -stable vector bundle of $\text{rk}(E_1) = r_1$ and $\deg(E_1) = d_1$, where $\deg(E_1) = (c_1(E_1), H)/(H^2)$. Let E be a μ -stable sheaf of $\text{rk}(E) = lr$ and $\deg(E) = ld$. Then the non-trivial extension*

$$0 \rightarrow E_1 \rightarrow E' \rightarrow E \rightarrow 0 \quad (1.2)$$

is a μ -stable sheaf.

Lemma 1.5 ([Y5, Lem. 4.4]). *Let v be an arbitrary Mukai vector of $\text{rk } v > 0$. Let $\mathcal{M}_H(v)^{\mu ss}$ be the moduli stack of μ -semi-stable sheaves E of $v(E) = v$, and $\mathcal{M}_H(v)^{p\mu ss}$ the closed substack of $\mathcal{M}_H(v)^{\mu ss}$ consisting of properly μ -semi-stable sheaves. We assume that $\langle v^2 \rangle / 2 \geq l^2$. Then*

$$\text{codim } \mathcal{M}_H(v)^{p\mu ss} \geq \langle v^2 \rangle / 2l - l + 1. \quad (1.3)$$

In particular, if $\mathcal{M}_H(v)^{\mu ss}$ is not empty, then there is a μ -stable sheaf E of $v(E) = v$.

Proof. Since we need the proof of this lemma, we shall give an outline of the proof. For more details, see [Y5, sect. 5.3]. By Mukai [Mu1], we get that

$$\dim \mathcal{M}_H(v)^{\mu ss} \geq (\langle v^2 \rangle + 2) - 1. \quad (1.4)$$

We shall show that

$$\dim \mathcal{M}_H(v)^{pss} \leq (\langle v^2 \rangle + 1) - (\langle v^2 \rangle / 2l - l + 1). \quad (1.5)$$

For this purpose, we shall estimate the moduli number of Jordan-Hölder filtrations. Let E be a μ -semi-stable sheaf of $v(E) = v$ and let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_t = E$ be a Jordan-Hölder filtration of E with respect to μ -stability. We set $E_i := F_i / F_{i-1}$. By using Lemma 5.1 in Appendix successively, we see that the moduli number of this filtration is bounded by

$$\sum_{i \leq j} (\dim \text{Ext}^1(E_j, E_i) - \dim \text{Hom}(E_j, E_i)) = -\chi(E, E) + \sum_{i > j} \chi(E_j, E_i) + \sum_{i < j} \dim \text{Ext}^2(E_j, E_i) + t. \quad (1.6)$$

We set $v(E) := lr + l\xi + a\omega$ and $v(E_i) := l_i r + l_i \xi + a_i \omega$, where $\xi \in \text{NS}(X)$. Since $\langle v(E_i), v(E_j) \rangle = l_i l_j (\xi^2) - r(l_i a_j + l_j a_i)$, we see that

$$\sum_{i > j} \chi(E_j, E_i) = -\sum_{i > j} \langle v(E_j), v(E_i) \rangle = -\sum_i \frac{(l - l_i) \langle v(E_i)^2 \rangle}{2l_i}. \quad (1.7)$$

We set $\max_i \{l_i\} = (l - k)$. Let i_0 be an integer such that $\langle v(E_{i_0})^2 \rangle \geq 0$. Since $\sum_i l_i = l$, we obtain that $t \leq k + 1$. Since $l - l_i - k \geq 0$ and $\langle v(E_i)^2 \rangle \geq -2$, we get that

$$\begin{aligned} \sum_{i > j} \langle v(E_j), v(E_i) \rangle &= k \sum_i \frac{\langle v(E_i)^2 \rangle}{2l_i} + \sum_i \frac{(l - l_i - k) \langle v(E_i)^2 \rangle}{2l_i} \\ &\geq k \frac{\langle v(E)^2 \rangle}{2l} - \sum_{i \neq i_0} (l - l_i - k) \\ &\geq k \frac{\langle v(E)^2 \rangle}{2l} - (l - 1 - k)k. \end{aligned}$$

If $r > 1$ or $l_i > 1$ for some i , then for a general filtration, there are E_i and E_j such that $\text{Ext}^2(E_j, E_i) = 0$. Therefore we get that $\sum_{i < j} \dim \text{Ext}^2(E_j, E_i) \leq (k + 1)k/2 - 1$ for a general filtration. Then the moduli number of these filtrations is bounded by

$$\begin{aligned} \langle v^2 \rangle - k \frac{\langle v^2 \rangle}{2l} + (l - 1 - k)k + \frac{(k + 1)k}{2} - 1 + t &\leq (\langle v^2 \rangle + 1) - \left(k \frac{\langle v^2 \rangle}{2l} - lk + \frac{k(k - 1)}{2} + 1 \right) \\ &\leq (\langle v^2 \rangle + 1) - \left(\frac{\langle v^2 \rangle}{2l} - l + 1 \right). \end{aligned}$$

Therefore we get a desired estimate for this case. In particular, our lemma holds for the case where $\text{rk } v / \ell(v) > 1$. For the case where $l_i = 1$ for all i and $r = 1$, see [Y5, sect. 5.3]. \square

1.3. Estimate on properly semi-stable sheaves.

Lemma 1.6. (1) Let E be a stable sheaf and F a semi-stable sheaf such that $v(F)/\text{rk } F = v(E)/\text{rk } E$. Then $\text{Hom}(E, F) \otimes E \rightarrow F$ is injective. In particular $\dim \text{Hom}(E, F) \leq \text{rk } F / \text{rk } E$.
(2) Let E be a μ -stable sheaf and F a μ -semi-stable sheaf such that $c_1(F)/\text{rk } F = c_1(E)/\text{rk } E$. Then $\text{Hom}(E, F) \otimes E \rightarrow F$ is injective. In particular $\dim \text{Hom}(E, F) \leq \text{rk } F / \text{rk } E$.

Lemma 1.7. Let v be a Mukai vector of $\langle v^2 \rangle > 0$ (we don't assume the primitivity of v). We set

$$\mathcal{M}(v)^{pss} := \{E \in \mathcal{M}(v)^{ss} \mid E \text{ is properly semi-stable}\}. \quad (1.8)$$

Then $\dim \mathcal{M}(v)^{pss} \leq \langle v^2 \rangle$. In particular, if $\mathcal{M}(v)^{ss} \neq \emptyset$, then $\mathcal{M}(v)^s \neq \emptyset$ and $\dim \mathcal{M}(v)^{ss} = \langle v^2 \rangle + 1$.

Proof. We set $v = lv'$, where v' is a primitive Mukai vector. We shall prove this lemma by induction on l . Let E_1 be a stable sheaf of $v(E_1) = l_1 v'$ and E_2 a semi-stable sheaf of $v(E_2) = l_2 v'$, where $l_1 + l_2 = l$. By induction hypothesis, $\dim \mathcal{M}^{ss}(v_i) = \langle v_i^2 \rangle + 1$, $i = 1, 2$. We shall estimate the dimension of the substack $J(v_1, v_2)$ whose element E fits in an extension

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0. \quad (1.9)$$

By Lemma 1.6, $\dim \text{Ext}^2(E_2, E_1) = \dim \text{Hom}(E_1, E_2) \leq l_2/l_1$. Moreover if E_1 is general, then $\text{Hom}(E_1, E_2) = 0$. Hence by Lemma 5.1 in Appendix, we get

$$\begin{aligned} \dim J(v_1, v_2) &\leq \dim \mathcal{M}(v_1)^{ss} + \dim \mathcal{M}(v_2)^{ss} + \langle v_1, v_2 \rangle + \max\{l_2/l_1 - 1, 0\} \\ &= \langle v_1^2 \rangle + \langle v_2^2 \rangle + 2 + \langle v_1, v_2 \rangle + \max\{l_2/l_1 - 1, 0\} \\ &= (\langle v^2 \rangle + 1) - (\langle v_1, v_2 \rangle - \max\{l_2/l_1, 1\}). \end{aligned} \quad (1.10)$$

Since

$$\langle v_1, v_2 \rangle = l_1 \frac{\langle v_2^2 \rangle}{2l_2} + l_2 \frac{\langle v_1^2 \rangle}{2l_1} \geq l, \quad (1.11)$$

we get $\dim J(v_1, v_2) \leq \langle v^2 \rangle$. Therefore we get our lemma. \square

1.4. Semi-stable sheaves of isotropic Mukai vector.

Lemma 1.8. Let w be a primitive isotropic Mukai vector of $\text{rk } w > 0$. Then $\dim \mathcal{M}(lw)^{ss} = l$.

Proof. Let E be a semi-stable sheaf of $v(E) = lw$. We shall first prove that there are stable sheaves E_1, E_2, \dots, E_k of $v(E_i) = w$ such that

$$E \cong \bigoplus_{i=1}^k F_i, \quad (1.12)$$

where F_i are S -equivalent to $E_i^{\oplus n_i}$.

Proof of the claim: By the proof of Mukai [Mu2, Prop. 4.4] (Fourier-Mukai transform for $H^*(X, \mathbb{Q})$), there is an element E_1 of $M_H(w)$ such that $\text{Hom}(E_1, E) \neq 0$. By induction hypothesis, $E/E_1 \cong F_1 \oplus F_2 \oplus \dots \oplus F_k$, where F_1 is S -equivalent to $E_1^{\oplus n_1}$, $n_1 \geq 0$ and F_i , $i > 1$ are S -equivalent to $E_i^{\oplus n_i}$. Since $\text{Ext}^1(E_i, E_1) = 0$ for $i > 1$, $\text{Ext}^1(\bigoplus_{i>1} F_i, E_1) = 0$. Therefore $\bigoplus_{i>1} F_i$ is a direct summand of E , which implies our claim.

Let E_1 be a stable sheaf of $v(E_1) = w$. We set

$$\mathcal{J}(l, E_1) := \{E \in \mathcal{M}(lw)^{ss} \mid E \text{ is } S\text{-equivalent to } E_1^{\oplus l}\}. \quad (1.13)$$

This is a closed substack of $\mathcal{M}(lw)^{ss}$ (see Appendix 4.3). We shall next prove that

$$\dim \mathcal{J}(l, E_1) \leq -1. \quad (1.14)$$

Since E_1 is parametrized by the surface $M_H(w)$, (1.14) implies that

$$\dim \mathcal{M}(lw)^{ss} = l. \quad (1.15)$$

For more details, see Appendix 4.3. Proof of (1.14): We set

$$\mathcal{J}(l, E_1, n) := \{E \in \mathcal{J}(l, E_1) \mid \dim \text{Hom}(E_1, E) = n\}. \quad (1.16)$$

By upper semi-continuity of cohomologies, this is a locally closed substack of $\mathcal{J}(l, E_1)$. If $n = l$, then $\mathcal{J}(l, E_1, l) = \{E_1^{\oplus l}\}$ and it is a closed substack of $\mathcal{J}(l, E_1)$. For an element E of $\mathcal{J}(l, E_1, n)$, there is an exact sequence

$$0 \rightarrow \text{Hom}(E_1, E) \otimes E_1 \rightarrow E \rightarrow E' \rightarrow 0 \quad (1.17)$$

where $E' \in \mathcal{J}(l-n, E_1, n')$. The moduli number of E which fits in this type of extension is equal to $\dim \mathcal{J}(l-n, E_1, n') + nn' - n^2$. Indeed, $\text{Hom}(E_1, E) \otimes E_1 (\cong E_1^{\oplus n})$ belongs to $\mathcal{J}(n, E_1, n)$ and $\dim \mathcal{J}(n, E_1, n) = -n^2$. Hence by the proof of Lemma 5.1, we get the equality. Therefore we see that

$$\dim \mathcal{J}(l, E_1) = - \min_{\substack{l=n_1+n_2+\dots+n_s \\ n_1, n_2, \dots, n_s \geq 1}} \left(\sum_{i=1}^s n_i^2 - \sum_{i=1}^{s-1} n_i n_{i+1} \right) \leq -1. \quad (1.18)$$

□

Remark 1.1. If $M_H(w)$ has a universal family, then Fourier-Mukai transform is defined [Br]. Then $\mathcal{M}(lw)^{ss}$ is transformed to the stack of 0-dimensional sheaves on $M_H(w)$. In this case, by using [Y1, Thm. 0.4], we can get our lemma.

1.5. Lemma 0.4 and its extensions.

1.5.1. *Proof of Lemma 0.4.* Proof of (1): By our assumptions, we have

$$\frac{d_1}{r_1} = \frac{\deg E_1}{\text{rk } E_1} < \frac{\deg \varphi(E_1)}{\text{rk } \varphi(E_1)} \leq \frac{\deg E}{\text{rk } E} = \frac{d}{r}. \quad (1.19)$$

By Lemma 1.3, $\deg \varphi(E_1)/\text{rk } \varphi(E_1) = d/r$. Hence $\varphi(E_1)$ coincides with E except finite points of X . Thus φ is surjective in codimension 1. We shall next prove that $\ker \varphi$ is a μ -stable vector bundle. Since $r_1 - lr$ and $d_1 - ld$ are relatively prime, we shall prove that $\ker \varphi$ is μ -semi-stable. If $\ker \varphi$ is not μ -semi-stable, then there is a subsheaf I of $\ker \varphi$ such that I is μ -stable and $d_1/r_1 > \deg(I)/\text{rk}(I) > (d_1 - ld)/(r_1 - lr)$. Since $d/r > d_1/r_1$, we see that

$$\frac{1}{r(r_1 - lr)} > \frac{d}{r} - \frac{\deg(I)}{\text{rk}(I)} \geq \frac{1}{r \text{rk}(I)}. \quad (1.20)$$

Hence $\text{rk}(I) > r_1 - lr$, which is a contradiction. Therefore $\ker \varphi$ is a μ -stable vector bundle.

Proof of (2): If ϕ is not injective, then $(d_1 - ld)/(r_1 - lr) < \deg(\phi(E'))/\text{rk}(\phi(E')) < d_1/r_1$. Since $d/r > d_1/r_1$, we see that

$$\frac{1}{r(r_1 - lr)} > \frac{d}{r} - \frac{\deg(\phi(E'))}{\text{rk}(\phi(E'))} \geq \frac{1}{r \text{rk}(\phi(E'))}. \quad (1.21)$$

Hence $\text{rk}(\phi(E')) > r_1 - lr$, which is a contradiction. Thus ϕ is injective. Assume that E is not μ -semi-stable. Then there is a μ -stable quotient sheaf F of E such that $\deg F/\text{rk } F < \deg E/\text{rk } E$. Since F is also a quotient sheaf of E_1 , we have

$$\frac{d_1}{r_1} = \frac{\deg E_1}{\text{rk } E_1} < \frac{\deg F}{\text{rk } F} < \frac{\deg E}{\text{rk } E} = \frac{d}{r}. \quad (1.22)$$

By Lemma 1.3, we get $\text{rk } F \geq r_1 + r$, which is a contradiction. Therefore E is μ -semi-stable. □

Remark 1.2. In order to define $\deg(E)$ in Lemma 0.4, we assumed that $\text{NS}(X) = \mathbb{Z}H$. However $\deg(E)$ is still defined if $(c_1(E), H)|(H, D)$ for all $D \in \text{NS}(X)$. Hence Lemma 0.4 also holds under this assumption.

1.5.2. *Extensions of Lemma 0.4 and [Y5, Lem. 4.1].* We shall extend Lemma 0.4 and [Y5, Lem. 4.1]. Let (X, H) be a polarized smooth projective surface of $\text{NS}(X) = \mathbb{Z}H$.

Lemma 1.9. *Let E be a μ -semi-stable vector bundle of $\text{rk}(E) = lr$ and $\deg(E) = ld$ which is defined by a non-trivial extension*

$$0 \rightarrow F_1 \rightarrow E \rightarrow F_2 \rightarrow 0, \quad (1.23)$$

where F_1 and F_2 are μ -stable vector bundles of $\deg(F_1)/\text{rk}(F_1) = \deg(F_2)/\text{rk}(F_2) = d/r$. Let E_1 be a μ -stable vector bundle in Lemma 0.4. Let $\varphi : E_1 \rightarrow E$ be a non-trivial homomorphism. Then φ is surjective in codimension 1, or $\varphi(E_1)$ is a subsheaf of F_1 . In particular, if $\text{Hom}(E_1, F_1) = 0$, then $\ker \varphi$ is μ -stable.

Proof. We assume that φ is not surjective in codimension 1. By Lemma 1.3, $\deg(\varphi(E_1))/\text{rk}(\varphi(E_1)) = d/r$. Assume that $\varphi(E_1) \rightarrow F_2$ is not 0. Then the μ -stability of F_2 implies that it is surjective in codimension 1. By the μ -stability of F_1 , $F_1 \cap \varphi(E_1) = 0$. Thus we can regard $\varphi(E_1)$ as a subsheaf of F_2 . Let $e \in \text{Ext}^1(F_2, F_1)$ be the extension class of (1.23). By the homomorphism $\text{Ext}^1(F_2, F_1) \rightarrow \text{Ext}^1(\varphi(E_1), F_1)$, e goes to 0. Since F_1 is a vector bundle, $\text{Ext}^1(F_2/\varphi(E_1), F_1) = 0$. Hence $\text{Ext}^1(F_2, F_1) \rightarrow \text{Ext}^1(\varphi(E_1), F_1)$ is injective. Thus we get that $e = 0$, which is a contradiction. Hence $\varphi(E_1) \rightarrow F_2$ is a 0-map, which means that $\varphi(E_1) \subset F_1$. The last assertion follows from the proof of Lemma 0.4. □

Lemma 1.10. *Keep the notations in Lemma 1.9. Assume that $\text{Ext}^1(E_1, F_2) = 0$. Then a non-trivial extension of E by E_1 is μ -stable.*

Proof. Let E' be a non-trivial extension of E by E_1 .

$$0 \rightarrow E_1 \rightarrow E' \rightarrow E \rightarrow 0. \quad (1.24)$$

We shall prove that E' is μ -stable. We consider the following diagram which is induced by the extension (1.24).

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ & & F_2 & \xlongequal{\quad} & F_2 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E_1 & \longrightarrow & E' & \longrightarrow & E \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & E_1 & \longrightarrow & E'' & \longrightarrow & F_1 \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array} \quad (1.25)$$

By Serre duality, $\text{Ext}^1(F_2, E_1) = 0$. Hence $\text{Ext}^1(E, E_1) \rightarrow \text{Ext}^1(F_1, E_1)$ is injective. Thus the last horizontal sequence of (1.25) does not split. By Lemma 1.4, E'' is μ -stable. If the middle vertical sequence splits, then $E \cong F_1 \oplus F_2$, which is a contradiction. Hence E' is a non-trivial extension of F_2 by a locally free sheaf E'' . By the construction of E'' , the conditions in Lemma 1.4 hold. Hence applying Lemma 1.4 again, we see that E' is μ -stable. \square

1.5.3. *Extension of [Y5, Lem. 4.2].* Let (X, H) be a polarized K3 surface of $\text{Pic}(X) = \mathbb{Z}H$. Let E_1 be an exceptional vector bundle of $v(E_1) := r_1 + d_1H + a_1\omega$ and let $v = l(r + dH) + a\omega$ be a primitive Mukai vector of $dr_1 - d_1r = 1$. We set

$$M_H(v)_i^{\mu s} := \{E \in M_H(v)^{\mu s} \mid \dim \text{Hom}(E_1, E) = i - \langle v, v(E_1) \rangle\}. \quad (1.26)$$

Assume that $r_1 > lr$. Then we get the following estimate which is necessary for the condition section 0.2 (a).

Lemma 1.11. (1) *If $\langle v, v(E_1) \rangle < 0$, then*

$$\text{codim}_{M_H(v)^{\mu s}} M_H(v)_i^{\mu s} \geq -\langle v, v(E_1) \rangle + 1 \geq 2 \quad (1.27)$$

for $i \geq 1$.

(2) *If $\langle v, v(E_1) \rangle \geq 0$, then*

$$\text{codim}_{M_H(v)^{\mu s, \text{loc}}} (M_H(v)_i^{\mu s} \cap M_H(v)^{\mu s, \text{loc}}) \geq \langle v, v(E_1) \rangle + 1 \geq 1 \quad (1.28)$$

for $i > \langle v, v(E_1) \rangle$, where $M_H(v)^{\mu s, \text{loc}} = M_H(v)^{\mu s} \cap M_H(v)^{\text{loc}}$.

Proof. We set

$$N_i := \left\{ E_1 \subset E \mid \begin{array}{l} E \in M_H(u), \dim \text{Hom}(E_1, E) = i + 1 - \langle v, v(E_1) \rangle, \\ E/E_1 \text{ is a } \mu\text{-stable sheaf of } v(E/E_1) = v \end{array} \right\}, \quad (1.29)$$

where $u = v + v(E_1)$. Then we see that $\dim N_i \leq i - \langle v, v(E_1) \rangle + \dim M_H(u) = i + \langle v, v(E_1) \rangle + \dim M_H(v) - 2$. Let $\pi'_v : N_i \rightarrow M_H(v)^{\mu s}$ be the morphism sending $(E_1 \subset E) \in N_i$ to $E/E_1 \in M_H(v)^{\mu s}$. By Lemma 1.4, $\pi'_v(N_i) = M_H(v)_i^{\mu s}$ and $\pi'^{-1}_v(E/E_1)$ is isomorphic to the projective space $\mathbb{P}(\text{Ext}^1(E/E_1, E_1)^\vee)$. Hence we get that $\dim M_H(v)_i^{\mu s} = \dim N_i - (i - 1) \leq \dim M_H(v) + \langle v, v(E_1) \rangle - 1$. Thus (1) holds.

We next prove the second claim. Hence we assume that $i > \langle v, v(E_1) \rangle$. For $E \in M_H(v)_i^{\mu s} \cap M_H(v)^{\mu s, \text{loc}}$, we choose a homomorphism $\phi : E_1 \rightarrow E$. By Lemma 0.4, ϕ is surjective in codimension 1. We set $G := \text{coker}(E^\vee \rightarrow E_1^\vee)$. By Lemma 0.4, F is a stable sheaf of $v(G) = w := v(E_1)^\vee - v^\vee$. It is easy to see that $\dim \text{Hom}(E_1^\vee, G) = \dim \text{Ext}^1(E_1, E) + 1 = \langle v, v(E_1) \rangle + i + 1$. Hence $\phi : E_1 \rightarrow E$ is parametrized by an open subscheme of a projective bundle of dimension $(\langle v, v(E_1) \rangle + i)$ over the subscheme $M_H(w)_i$, where

$$\begin{aligned} M_H(w)_i &:= \{G \in M_H(w) \mid \dim \text{Hom}(E_1^\vee, G) = \langle v, v(E_1) \rangle + i + 1\} \\ &= \{G \in M_H(w) \mid \dim \text{Hom}(E_1^\vee, G) = \langle w, v(E_1)^\vee \rangle + (i - 1)\}. \end{aligned} \quad (1.30)$$

Thus we see that

$$\begin{aligned} \dim M_H(v)_i^{\mu s} \cap M_H(v)^{\mu s, \text{loc}} &\leq \langle w^2 \rangle + 2 + (\langle v, v(E_1) \rangle + i) - (i - 1) \\ &= \langle v^2 \rangle + 2 - (\langle v, v(E_1) \rangle + 1). \end{aligned} \quad (1.31)$$

□

2. CASE A

2.1. **Estimate.** In this section, we fix a primitive Mukai vector $r + \xi, \xi \in \text{NS}(X)$. We assume that

$$((\xi^2) + 2)/2r \notin \mathbb{Z}. \quad (2.1)$$

We shall prove Theorem 0.1 for a primitive Mukai vector $v := l(r + \xi) + a\omega \in H^*(X, \mathbb{Z})$. We shall first estimate the dimensions of various locally closed substacks of $\mathcal{M}(v)$.

Lemma 2.1. *If $\mathcal{M}(v)^{\mu ss} \neq \emptyset$, then $\langle v^2 \rangle \geq 0$. If the equality holds, then $\mathcal{M}(v)^{\mu ss} = \mathcal{M}(v)^{ss}$.*

Proof. Let E be a μ -semi-stable sheaf of $v(E) = v$ and E is S -equivalent to $\bigoplus_{i=1}^s E_i$ with respect to μ -stability, where $E_i, 1 \leq i \leq s$ are μ -stable sheaves. We set

$$v(E_i) := l_i(r + \xi) + a_i\omega, 1 \leq i \leq s. \quad (2.2)$$

By our assumption (2.1), $\langle v(E_i)^2 \rangle = l_i(l_i(\xi^2) - 2ra_i) \neq -2$. Thus $\langle v(E_i)^2 \rangle \geq 0$ for all i . Since

$$\frac{\langle v^2 \rangle}{l} = \sum_{i=1}^s \frac{\langle v(E_i)^2 \rangle}{l_i}, \quad (2.3)$$

we get $\langle v^2 \rangle \geq 0$. If $\langle v^2 \rangle = 0$, then $\langle v(E_i)^2 \rangle = 0$ for all i . Since $\langle v(E_i)^2 \rangle / \text{rk}(E_i)^2 = (\xi^2) - 2a_i/r l_i$ and $\chi(E_i) / \text{rk}(E_i) = 1 + a_i/r l_i$, we see that $\chi(E_i) / \text{rk}(E_i) = \chi(E) / \text{rk}(E)$ for all i . Thus E is semi-stable. □

Corollary 2.2. *If $\langle v^2 \rangle = 0$, then $\mathcal{M}(v)^{\mu ss}$ consists of locally free sheaves.*

Definition 2.1. Let $w = l_0(r + \xi) + a_0\omega$ be the primitive Mukai vector such that $\langle w^2 \rangle = 0$.

By Lemma 2.1 and Corollary 2.2, $M_H(w)$ consists of μ -stable locally free sheaves.

Lemma 2.3. (1)

$$\dim(\mathcal{M}(v)^{\mu ss} \setminus \mathcal{M}(v)^{ss}) \leq \langle v^2 \rangle. \quad (2.4)$$

(2) Assume that $\langle v^2 \rangle > 0$. Then

$$\dim(\mathcal{M}(v)^{\mu ss} \setminus \mathcal{M}(v)^s) \leq \langle v^2 \rangle. \quad (2.5)$$

In particular, if $\mathcal{M}(v)^{\mu ss} \neq \emptyset$, then $\mathcal{M}(v)^s \neq \emptyset$ and $\dim \mathcal{M}(v)^{\mu ss} = \langle v^2 \rangle + 1$.

Proof. By Lemma 1.7, it is sufficient to prove (1). Let F be a μ -semi-stable sheaf of $v(F) = v$. We assume that F is not stable. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \quad (2.6)$$

be the Harder-Narasimhan filtration of F . We set

$$v_i := v(F_i/F_{i-1}) = l_i(r + \xi) + a_i\omega, 1 \leq i \leq s. \quad (2.7)$$

Since $\chi(F_i/F_{i-1}) / \text{rk}(F_i/F_{i-1}) > \chi(F_{i+1}/F_i) / \text{rk}(F_{i+1}/F_i)$, we get that

$$\frac{a_0}{l_0} \geq \frac{a_1}{l_1} > \frac{a_2}{l_2} > \cdots > \frac{a_s}{l_s}. \quad (2.8)$$

Let $\mathcal{F}^{HN}(v_1, v_2, \dots, v_s)$ be the substack of $\mathcal{M}(v)^{\mu ss}$ whose element E has the Harder-Narasimhan filtration of the above type. We shall prove that $\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \leq \langle v^2 \rangle$. Since $\text{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0$ for $i < j$, Lemma 5.2 in Appendix implies that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) = \sum_{i=1}^s \dim \mathcal{M}(v_i)^{ss} + \sum_{i < j} \langle v_j, v_i \rangle. \quad (2.9)$$

For $i < j$, by using Lemma 2.1 and (2.8), we see that

$$\begin{aligned} \langle v_i, v_j \rangle &= l_i l_j (\xi^2) - (l_i a_j + l_j a_i) r \\ &= l_i l_j (\xi^2) - 2l_j a_i r + (a_i l_j - a_j l_i) r \\ &= l_j (l_i (\xi^2) - 2a_i r) + (a_i l_j - a_j l_i) r \\ &\geq (a_i l_j - a_j l_i) r \geq r \geq 2, \end{aligned} \quad (2.10)$$

where the inequality $r \geq 2$ comes from our assumption (2.1). Hence if $\langle v_i^2 \rangle > 0$ for all i , then, by using Lemma 1.7, we see that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \leq (\langle v^2 \rangle + 1) - \left(\sum_{i < j} \langle v_i, v_j \rangle - s + 1 \right) \leq \langle v^2 \rangle. \quad (2.11)$$

Assume that $\langle v_i^2 \rangle = 0$, i.e. $v_i = l'_i w$, $l_i \in \mathbb{Z}$. Then $i = 1$ and $(a_1 l_j - a_j l_1)$ is divisible by l'_i . Hence

$$\begin{aligned} \langle v_1, v_j \rangle - l'_1 &= l'_1 (\langle w, v_j \rangle - 1) \\ &\geq l'_1 (r - 1) > 0. \end{aligned} \quad (2.12)$$

In this case, by using Lemma 1.8, we see that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \leq (\langle v^2 \rangle + 1) - \left(\sum_{i < j} \langle v_i, v_j \rangle - (l'_1 + s - 1) + 1 \right) \leq \langle v^2 \rangle. \quad (2.13)$$

Hence we get our lemma. \square

Proposition 2.4. *Assume that $\langle v^2 \rangle > 0$. Then*

$$\dim(\mathcal{M}(v)^s \setminus \mathcal{M}(v)^{\mu s}) \leq \langle v^2 \rangle. \quad (2.14)$$

In particular, $\mathcal{M}(v)^{\mu s} \neq \emptyset$, if $\mathcal{M}(v)^s \neq \emptyset$.

Proof. Let E be a stable sheaf and E_1 be a μ -stable subsheaf of E such that E/E_1 is torsion free. We set

$$\begin{aligned} v_1 &:= v(E_1) = l_1(r + \xi) + a_1 \omega, \\ v_2 &:= v(E/E_1) = l_2(r + \xi) + a_2 \omega. \end{aligned} \quad (2.15)$$

Since $\chi(E_1)/\text{rk } E_1 < \chi(E)/\text{rk } E$, we get $\langle v(E_1)^2 \rangle > 0$ and

$$\frac{a_1}{l_1} < \frac{a_2}{l_2}. \quad (2.16)$$

Let $J(v_1, v_2)$ be the substack of $\mathcal{M}(v)^s$ consisting of E which has a subsheaf $F_1 \subset E$. By using Lemma 5.1 in Appendix, we shall estimate $\dim J(v_1, v_2)$. By Lemma 1.6, $\dim \text{Hom}(E_1, E/E_1) \leq l_2/l_1$, and if E_1 is general, then $\text{Hom}(E_1, E/E_1) = 0$. We shall bound the dimension of the substack

$$\mathcal{N}(v_1, v_2) := \{(E_1, E_2) \in \mathcal{M}(v_1)^{\mu s s} \times \mathcal{M}(v_2)^{\mu s s} \mid \dim \text{Hom}(E_1, E_2) \neq 0\}. \quad (2.17)$$

For a fixed $E_2 \in \mathcal{M}(v_2)^{ss}$,

$$\#\{E_1^{\vee\vee} \mid E_1 \in \mathcal{M}(v_1)^{\mu s}, \text{Hom}(E_1, E_2) \neq 0\} < \infty. \quad (2.18)$$

Hence, by using [Y1, Thm. 0.4], we see that

$$\dim\{E_1 \in \mathcal{M}(v_1)^{\mu s} \mid \text{Hom}(E_1, E_2) \neq 0\} \leq \dim \mathcal{M}(v_1)^{\mu s} - 2 - (\text{rk } v_1 - 1). \quad (2.19)$$

Thus $\dim \mathcal{N}(v_1, v_2) \leq \dim \mathcal{M}(v_1)^{\mu s s} + \dim \mathcal{M}(v_2)^{\mu s s} - 3$. Moreover, taking (1.12) into account, if $l_1 \neq l_0$ and $v_2 = l'_2 w$, $l'_2 \in \mathbb{Z}$, then we get $\mathcal{N}(v_1, v_2) = \emptyset$.

If $\langle v_2^2 \rangle > 0$, then Lemma 2.3 implies that $\dim \mathcal{M}(v_2)^{\mu s s} = \langle v_2^2 \rangle + 1$. Hence Lemma 5.1 implies that

$$\begin{aligned} \dim \mathcal{M}(v)^s - \dim J(v_1, v_2) &= \min \left(\langle v_1, v_2 \rangle - \frac{l_2}{l_1} + 2, \langle v_1, v_2 \rangle - 1 \right) \\ &= l_1 \frac{\langle v_2^2 \rangle}{2l_2} + l_2 \frac{\langle v_1^2 \rangle}{2l_1} - \max \left(\frac{l_2}{l_1} - 2, 1 \right) > 0. \end{aligned} \quad (2.20)$$

We next treat the case where $\langle v_2^2 \rangle = 0$. Then $v_2 = l'_2 w$, $l'_2 \in \mathbb{Z}$. By Lemma 2.3 (1) and Lemma 1.8, $\dim \mathcal{M}(v_2)^{\mu s s} = \langle v_2^2 \rangle + l'_2$. If $l_1 = l_0$, then $l_2/l_1 = l'_2$. So we see that

$$\begin{aligned} \dim \mathcal{M}(v)^s - \dim J(v_1, v_2) &= \min(l'_2(\langle v_1, w \rangle - 1 - 1) + 3, l'_2(\langle v_1, w \rangle - 1)) \\ &= \min\{l'_2((a_0 l_1 - a_1 l_0)r - 2) + 3, l'_2((a_0 l_1 - a_1 l_0)r - 1)\} > 0. \end{aligned} \quad (2.21)$$

\square

Remark 2.1. By the proof of Lemma 2.3 and Proposition 2.4, we see that

$$\text{codim}_{\mathcal{M}(v)^{\mu s s}}(\mathcal{M}(v)^{\mu s s} \setminus \mathcal{M}(v)^{\mu s}) \geq 2 \text{ for } r \geq 3. \quad (2.22)$$

Moreover if $r = 2$, then the general member E of $\mathcal{M}(v)^s \setminus \mathcal{M}(v)^{\mu s}$ fits in the following exact sequence

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0 \quad (2.23)$$

where E_1 is a μ -stable vector bundle and E_2 is a μ -stable vector bundle of $v(E_2) = w$.

Indeed, if $\text{codim}_{\mathcal{M}(v)^s}(\mathcal{M}(v)^s \setminus \mathcal{M}(v)^{\mu s}) = 1$, then (1) $\langle v_2^2 \rangle > 0$, $l_1 = l_2 = \langle v_1^2 \rangle/2 = \langle v_2^2 \rangle/2 = 1$, or (2) $r = 2$, $l'_2 = 1$ and $a_0 l_1 - a_1 l_0 = 1$. By (2.16), case (1) does not occur.

2.2. Proof of Theorem 0.1 for case A.

2.2.1. *The case of $r > 2$.* In the same way as in the proof of Theorem 1.1, we shall prove Theorem 0.1 if $r > 2$. Let $v = l(r + \xi) + a\omega$ be a primitive Mukai vector on a K3 surface X such that $r + \xi, \xi \in \text{NS}(X)$ is primitive. We claim that we can find a primitive Mukai vector $v' = l(r + d'H') + a'\omega$ on a polarized K3 surface (X', H') of $\text{Pic}(X') = \mathbb{Z}H'$ and an exceptional vector bundle G of $v(G) := v_1 = r_1 + d_1H' + a_1\omega'$ such that (1) $r + d'H'$ is primitive, (2) $\langle v^2 \rangle = \langle (v')^2 \rangle$, (3) $a \equiv a' \pmod{l}$, (4) $d'r_1 - d_1r = 1$, (5) $r_1 - lr \geq 2$ and (6) $\langle v(G), v' \rangle = -1$, where ω' is the fundamental class of X' .

Proof of the claim: We choose integers r_1, d_1 and d' such that $ar_1 \equiv 1 \pmod{l}$ and $(r_1, r) = 1$. We can easily choose such an integer r_1 of $r_1 - lr \geq 2$. Then we can choose d' and d_1 of $d'r_1 - d_1r = 1$. Let q be an integer of $ar_1 + ql = 1$. We set $k(s) := r_1(qr + r_1s) - r^2, s = (\xi^2)/2 \in \mathbb{Z}$. Then $k(s) = (r_1\langle v^2 \rangle/2l + r)r_1/l - r^2$. Since $\langle v^2 \rangle \geq 0$, $k(s) \geq rr_1^2/l - r^2 > 0$. Let (X', H') be a polarized K3 surface such that $\text{Pic}(X') = \mathbb{Z}H'$ and $(H')^2 = 2k(s)$. We set

$$\begin{cases} v' = lr + ld'H' + \{l((1 + d'r_1)d_1s + d'^2qr_1 - rd'^2) + a\}\omega', \\ v_1 = r_1 + d_1H' + \{r_1(-d'^2 + d_1^2s) + d_1^2rq + 2d'\}\omega'. \end{cases} \quad (2.24)$$

Then we see that

$$\begin{cases} \langle v_1^2 \rangle = -2, \\ \langle v'^2 \rangle = 2l(ls - ra) = \langle v^2 \rangle, \\ \langle v_1, v' \rangle = -1. \end{cases} \quad (2.25)$$

Since r_1 and d_1 are relatively prime, Theorem 1.1 implies that there is an exceptional vector bundle G of $v(G) = v_1$. Then G and v' satisfy our claims.

We shall consider reflection defined by G . We note that

$$w := -R_{v_1}(v')^\vee = (r_1 - lr) - (d_1 - ld')H' + (a_1 - a')\omega'. \quad (2.26)$$

Since $r_1 - lr$ and $d_1 - ld'$ are relatively prime, Theorem 1.1 for a Mukai vector w of $\ell(w) = 1$ implies that $M_{H'}(w) \neq \emptyset$ and Theorem 0.1 holds for this space. Since $r_1 - lr \geq 2$ and $M_{H'}(w)$ consists of μ -stable sheaves, [Y1, Thm. 0.4] implies that there is a μ -stable vector bundle E of $v(E) = w$. By (6), we see that $\chi(G^\vee, E) = -\langle v(G)^\vee, w \rangle = 1$. Hence there is a non-trivial homomorphism $\phi : G^\vee \rightarrow E$. By using Lemma 0.4 (2), we see that $\text{coker}(\phi^\vee)$ is a μ -semi-stable sheaf of $v(\text{coker}(\phi^\vee)) = v'$. Then Lemma 2.3 and Proposition 2.4 imply that $M_{H'}(v')^{\mu s} \neq \emptyset$. Applying Lemma 0.4 and Lemma 1.11, we get a birational map

$$M_{H'}(v') \cdots \rightarrow M_{H'}(w) \quad (2.27)$$

sending $F \in M_{H'}(v')_0^{\mu s} \cap M_{H'}(v')^{\text{loc}}$ to $\text{coker}(F^\vee \rightarrow G^\vee)$, which means that Theorem 0.1 (1) and (2-1) hold for $M_{H'}(v')$. Then Proposition 1.2 implies that Theorem 0.1 (1) and (2-1) also hold for $M_H(v)$, where H is a general ample divisor on X .

Moreover by Remark 2.1, we can naturally identify $H^2(M_H(v'), \mathbb{Z})$ with $H^2(M_H(w), \mathbb{Z})$. Let $\mathcal{F}^H : H^*(X, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$ be an isomerty of Mukai lattice defined by $\mathcal{F}^H(x) = R_{v_1}(x)^\vee$. Then it is easy to see that the following diagram is commutative (see the computation in [Y5, 2.4]).

$$\begin{array}{ccc} v'^\perp & \xrightarrow{\mathcal{F}^H} & w^\perp \\ \theta_{v'} \downarrow & & \downarrow \theta_w \\ H^2(M_{H'}(v'), \mathbb{Z}) & \xlongequal{\quad} & H^2(M_{H'}(w), \mathbb{Z}) \end{array}$$

Hence (2-2) also holds if $r > 2$. □

2.2.2. *The case of $r = 2$.* We next treat the case where $r = 2$. It is sufficient to extend the birational map (2.27) to a general member $E \in M_{H'}(v')$ which fits in (2.23). Let G be the exceptional vector bundle in 2.2.1. By using Lemma 1.9 and 1.10, we can prove the following: Assume that $\langle v', v(G) \rangle = -1$. Then, for a general member E which fits in the exact sequence (2.23),

- (1) $\text{Ext}^1(G, E) = 0$,
- (2) $\phi : \text{Hom}(G, E) \otimes G \rightarrow E$ is surjective in codimension 1 and $\ker \phi$ is stable.

Proof. Since $\langle (v')^2 \rangle > 0$, we can write $v' = xw - y\omega$, where $x, y \in \mathbb{Q}$ and $x, y > 0$. Since $\langle v', v(G) \rangle = x\langle w, v(G) \rangle + y\text{rk}(G) = -1$, $\langle w, v(G) \rangle < 0$. Hence $\langle v(E_1), v(G) \rangle \geq 0$ and $\langle v(E_2), v(G) \rangle < 0$. Applying Lemma 1.11, we may assume that $\text{Hom}(G, E_1) = \text{Ext}^1(G, E_2) = 0$. By Lemma 1.10, we can use the same argument as in the proof of Lemma 1.11 (1). Hence (1) holds. Applying Lemma 1.9, we get (2). □

Therefore the dual of $\phi : E^\vee \rightarrow G^\vee$ is injective and the cokernel is stable. Thus the reflection induces a birational map $M_H(v) \setminus Z \rightarrow M_H(v(G)^\vee - v^\vee)$ such that $\text{codim}_{M_H(v)} Z \geq 2$. Therefore Theorem 0.1 also holds for this case. □

Remark 2.2. Under the assumptions for case A, the following holds.

- (1) $M_H(v) \neq \emptyset$ if and only if $\langle v^2 \rangle \geq 0$.
- (2) $M_H(v)^{\mu s} \neq \emptyset$ if and only if $\langle v^2 \rangle \geq 0$.

In particular, $M_H(v)$ contains μ -stable vector bundles if $M_H(v) \neq \emptyset$.

3. CASE B

In this section, we assume that there is a (-2) vector v_0 of the form $v_0 = r + \xi + b\omega$, $b \in \mathbb{Z}$. Let E_0 be the element of $M_H(v_0)$. We shall prove Theorem 0.1 for a primitive Mukai vector $v := lv_0 - a\omega$. If $\langle v^2 \rangle > 2l^2$, then the same proof in section 2.2.1 works, because of Lemma 1.5. Hence we may assume that $\langle v^2 \rangle \leq 2l^2$.

3.1. The case of $\langle v^2 \rangle < 2l^2$. We first treat the case where $\langle v^2 \rangle < 2l^2$. Clearly $\langle v^2 \rangle \geq -2$, if $M_H(v) \neq \emptyset$. If $\langle v^2 \rangle = -2$, then $2l(a \operatorname{rk} v_0 - l) = -2$. Hence we get $l = 1$. This case is covered in Theorem 1.1. So we assume that $\langle v^2 \rangle \geq 0$, that is

$$l \leq a \operatorname{rk} v_0 < 2l. \quad (3.1)$$

Based on the next key lemma, we shall prove Theorem 0.1 in 3.1.2 and 3.1.3.

Lemma 3.1. $\operatorname{codim}_{M_H(v)}(M_H(v) \setminus M_H(v)^{\operatorname{loc}}) \geq 2$ unless

$$v = \begin{cases} (\operatorname{rk} v_0)v_0 - \omega, \\ lv_0 - (l+1)\omega, \operatorname{rk} v_0 = 1. \end{cases} \quad (3.2)$$

3.1.1. Proof of Lemma 3.1. We set

$$\mathcal{M}(v, n)^s := \{E \in \mathcal{M}(v)^s \mid \dim(E^{\vee\vee}/E) = n\}. \quad (3.3)$$

For $E \in \mathcal{M}(v, n)^s$, let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E^{\vee\vee} \quad (3.4)$$

be the Harder-Narasimhan filtration of $E^{\vee\vee}$. We set

$$v_i := v(F_i/F_{i-1}) = l_i v_0 - a_i \omega, 1 \leq i \leq s. \quad (3.5)$$

We shall estimate the codimension of the substack

$$\mathcal{M}(v, n; v_1, v_2, \dots, v_s) := \{E \in \mathcal{M}(v, n)^s \mid E^{\vee\vee} \in \mathcal{F}^{HN}(v_1, v_2, \dots, v_s)\}, \quad (3.6)$$

where $\mathcal{F}^{HN}(v_1, v_2, \dots, v_s)$ is defined as in the proof of Lemma 2.3. We divide our consideration into two cases

- (I) $s \geq 2$, that is, $E^{\vee\vee}$ is not semi-stable.
- (II) $s = 1$, that is, $E^{\vee\vee}$ is semi-stable.

Case (I). (I-a) If $\langle v_1^2 \rangle \geq 0$, then we can use almost the same arguments as in the proof of Lemma 2.3. The difference comes from the inequality $r \geq 2$ which was used in (2.10) and (2.12). Thus

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \leq \langle v(E^{\vee\vee})^2 \rangle + 1. \quad (3.7)$$

The equality holds only if $r = 1$ and $\langle v_1^2 \rangle = 0$. Since F_s is locally free, F_1 is also locally free. On the other hand, if $r = 1$ and $\langle v_1^2 \rangle = 0$, then $\mathcal{M}(v_1)^{ss}$ consists of non-locally free sheaves (cf. Lemma 1.8). Therefore the equality does not hold. Hence, by using [Y1, Thm. 0.4], we see that

$$\begin{aligned} \dim \mathcal{M}(v, n; v_1, v_2, \dots, v_s) &< \langle (v - n\omega)^2 \rangle + 1 + n(\operatorname{rk} v + 1) \\ &\leq \langle v^2 \rangle + 1 - n(\operatorname{rk} v - 1). \end{aligned} \quad (3.8)$$

Thus we get a desired estimate

$$\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n; v_1, v_2, \dots, v_s) \geq 2. \quad (3.9)$$

(I-b) We assume that $\langle v_1^2 \rangle < 0$. Then $F_1 = E_0^{\oplus l_1}$, which implies that $\dim \mathcal{M}(v_1)^{ss} = -l_1^2 = \langle v_1^2 \rangle + l_1^2$. For convenience sake, we set

$$v'_2 := \sum_{i=2}^s v_i = l'_2 v_0 - a'_2 \omega. \quad (3.10)$$

(I-b-1) We first assume that

$$\langle (v'_2)^2 \rangle = 2l'_2(a'_2 \operatorname{rk} v_0 - l'_2) > 0. \quad (3.11)$$

Then, since $\langle v_i^2 \rangle \geq 0$ for all $i \geq 2$, $\dim \mathcal{F}^{HN}(v_2, \dots, v_s) \leq \langle (v'_2)^2 \rangle + 1$. Thus we get

$$\langle v(E^{\vee\vee})^2 \rangle + 1 - \dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \geq \langle v_1, v'_2 \rangle - l_1^2. \quad (3.12)$$

We shall prove that

$$\langle v_1, v'_2 \rangle - l_1^2 + n(l \operatorname{rk} v_0 - 1) \geq 2. \quad (3.13)$$

Proof. Since E is stable, $F_1 \cap E$ satisfies that

$$\frac{\chi(F_1 \cap E)}{\operatorname{rk} F_1} < \frac{\chi(E)}{\operatorname{rk} E}. \quad (3.14)$$

Since $\chi(F_1 \cap E) \geq \chi(F_1) - n$ and $v_1 = l_1 v_0$ (i.e. $a_1 = 0$), we see that

$$nl'_2 - a'_2 l_1 > 0. \quad (3.15)$$

By our assumption (3.1), $(a'_2 + n) \operatorname{rk} v_0 < 2(l_1 + l'_2)$. By using (3.11), we see that

$$n \leq \frac{2l_1 + l'_2 - 1}{\operatorname{rk} v_0}. \quad (3.16)$$

By using (3.11) and (3.15), we see that

$$nl_1 \operatorname{rk} v_0 \geq l_1^2 + 1. \quad (3.17)$$

Assume that $\operatorname{rk} v_0 \geq 2$. Then (3.16) implies that $n < l_1 + l'_2/2$. By using (3.11) and (3.17), we see that

$$\begin{aligned} \langle v_1, v'_2 \rangle - l_1^2 + n(l \operatorname{rk} v_0 - 1) &= l_1(-2l'_2 + a'_2 \operatorname{rk} v_0) + n((l_1 + l'_2) \operatorname{rk} v_0 - 1) - l_1^2 \\ &= -l_1 l'_2 + l_1(a'_2 \operatorname{rk} v_0 - l'_2) + nl_1 \operatorname{rk} v_0 + l_1 l'_2 + l'_2(n \operatorname{rk} v_0 - l_1) - n - l_1^2 \\ &\geq l_1(a'_2 \operatorname{rk} v_0 - l'_2) + l'_2(n \operatorname{rk} v_0 - l_1) + 1 - n \\ &\geq l_1 + l'_2 + 1 - n > 1. \end{aligned} \quad (3.18)$$

Assume that $\operatorname{rk} v_0 = 1$. Then similar computations work if $a'_2 \operatorname{rk} v_0 - l'_2 > 1$. So we assume that $\operatorname{rk} v_0 = a'_2 \operatorname{rk} v_0 - l'_2 = 1$. Then (3.17) implies that $n - l_1 > 0$. Hence we see that

$$l_1(-2l'_2 + a'_2 \operatorname{rk} v_0) + n((l_1 + l'_2) \operatorname{rk} v_0 - 1) - l_1^2 = (l_1 + l'_2 - 1)(n - l_1) \geq (l - 1) \geq 1. \quad (3.19)$$

If the equality holds, then $l = 2$ and $n - l_1 = 1$. In this case, we get that $l_1 = l'_2 = 1$ and $n = 2$. By (3.15), we get a contradiction. Thus the left hand side of (3.19) is greater than or equal to 2. Therefore (3.13) holds. \square

By (3.13) and [Y1, Thm. 0.4], we get a desired estimate

$$\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n; v_1, v_2, \dots, v_s) \geq 2. \quad (3.20)$$

(I-b-2) We next assume that

$$\langle (v'_2)^2 \rangle = 2l'_2(a'_2 \operatorname{rk} v_0 - l'_2) = 0. \quad (3.21)$$

Then $s = 2$ and $\langle v_0, v'_2 \rangle = -l'_2$. Since $\operatorname{Hom}(E_0, F_2/F_1) = 0$, $\dim \operatorname{Hom}(F_2/F_1, E_0) \geq l'_2$. By Lemma 1.6, coevaluation map

$$\psi : F_2/F_1 \rightarrow E_0 \otimes \operatorname{Hom}(F_2/F_1, E_0)^\vee \quad (3.22)$$

is surjective in codimension 1 (cf. Lemma 1.6). Therefore we get

- (i) $\dim \operatorname{Hom}(F_2/F_1, E_0) = l'_2$,
- (ii) $\dim \operatorname{Ext}^1(F_2/F_1, E_0) = 0$ and
- (iii) ψ is isomorphic in codimension 1.

By the definition of Harder-Narasimhan filtration, ψ is not isomorphic. Thus F_2/F_1 is not locally free. By (ii), $E^{\vee\vee} \cong F_1 \oplus F_2/F_1$, which contradicts the locally freeness of $E^{\vee\vee}$. Thus this case does not occur. By (I-a), (I-b-1) and (I-b-2), we get a desired bound for the case (I).

Case (II). We divide our consideration into three cases (II-a) $\langle v^2 \rangle > 0$, (II-b) $\langle v^2 \rangle = 0$ and (II-c) $\langle v^2 \rangle < 0$.

(II-a) If $\langle v_1^2 \rangle > 0$, then

$$\begin{aligned} \dim \mathcal{M}(v, n; v_1) &= \dim \mathcal{M}(v_1) + n(\operatorname{rk} v + 1) \\ &= \langle v_1^2 \rangle + 1 + n(\operatorname{rk} v + 1) \\ &= \langle v^2 \rangle + 1 - n(\operatorname{rk} v - 1). \end{aligned} \quad (3.23)$$

Hence if $\operatorname{rk} v \geq 3$, then $\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n; v_1) \geq 2$. If $\operatorname{rk} v = 2$, then the condition (3.1) implies that $a = 3$. Hence we get $\langle v_1^2 \rangle = \langle v^2 \rangle - 2n \operatorname{rk} v \leq 0$. Therefore this case does not occur.

(II-b) If $\langle v_1^2 \rangle = 0$, then the argument in (I-b-2) implies that $\mathcal{M}(v_1)^{ss}$ consists of non-locally free sheaves, which is a contradiction.

(II-c) We assume that $\langle v_1^2 \rangle < 0$, that is, $E^{\vee\vee} = E_0^{\oplus l}$. Then (3.1) implies that $n \operatorname{rk} v_0 - l \geq 0$.

(II-c-1) We first assume that $n \operatorname{rk} v_0 - l \geq 1$. Then we get

$$\begin{aligned} \operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n) &= (2nl \operatorname{rk} v_0 - 2l^2 + 1) - (n(l \operatorname{rk} v_0 + 1) - l^2) \\ &= n(l \operatorname{rk} v_0 - 1) - (l^2 - 1) \\ &\geq \frac{l+1}{\operatorname{rk} v_0} (l \operatorname{rk} v_0 - 1) - (l^2 - 1) \\ &= \frac{(l+1)(\operatorname{rk} v_0 - 1)}{\operatorname{rk} v_0} \geq 0, \end{aligned} \quad (3.24)$$

and the equality holds if and only if $\operatorname{rk} v_0 = 1$ and $n \operatorname{rk} v_0 - l = 1$. By the computation of (3.24), it is easy to show that $\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n) \geq 2$, if $\operatorname{rk} v_0 \geq 2$, or $n \operatorname{rk} v_0 - l \geq 2$.

(II-c-2) If $n \operatorname{rk} v_0 - l = 0$, then the primitivity of v implies that $n = 1$. Therefore we get a desired bound for the case (II).

By (I) and (II), we complete the proof of Lemma 3.1. \square

3.1.2. Components containing locally free sheaves. We shall prove Theorem 0.1 unless $v = (\operatorname{rk} v_0)v_0 - \omega$, or $\operatorname{rk} v_0 = 1$ and $v = lv_0 - (l+1)\omega$. For a locally free sheaf $E \in \mathcal{M}(v)^s$, we consider the dual of E . Since $\langle v_0, v \rangle = a \operatorname{rk} v_0 - 2l < 0$ and E is stable, $l' := \dim \operatorname{Ext}^2(E_0, E) \geq 2l - a \operatorname{rk} v_0 > 0$. By Serre duality, we get an exact sequence

$$0 \rightarrow (E_0^\vee)^{\oplus l'} \rightarrow E^\vee \rightarrow F \rightarrow 0, \quad (3.25)$$

where $F \in \mathcal{M}(v^\vee - l'v_0^\vee)^{\mu ss}$. Then we see that

$$\operatorname{Hom}(E_0^\vee, F) = \operatorname{Ext}^2(E_0^\vee, F) = 0. \quad (3.26)$$

Since $l' < l$, we see that

$$\begin{aligned} \langle v(F)^2 \rangle &= 2l(a \operatorname{rk} v_0 - l) - 2l'(l' - 2l + a \operatorname{rk} v_0) \\ &= 2(l - l')(a \operatorname{rk} v_0 - l + l') \\ &\geq 2(l - l')l > 2(l - l')^2. \end{aligned} \quad (3.27)$$

Hence, by Lemma 1.5, we get $\dim \mathcal{M}(v^\vee - l'v_0^\vee)^{\mu ss} = \langle (v - l'v_0)^2 \rangle + 1$. Taking into account (3.26), we see that the moduli number of E^\vee which fits in the exact sequence (3.25) is given by

$$\dim \mathcal{M}(v^\vee - l'v_0^\vee)^{\mu ss} + \dim \operatorname{Gr}(\langle v^\vee - l'v_0^\vee, v_0^\vee \rangle, l') = \langle v^2 \rangle + 1 - l'(l' + \langle v_0, v \rangle), \quad (3.28)$$

(cf. [Y5, Lem. 2.6]). We set

$$M_H(v)^* := \left\{ E \in M_H(v)^{loc} \left| \begin{array}{l} \operatorname{Ext}^1(E_0, E) = 0, \\ \operatorname{coker}(\operatorname{Hom}(E, E_0) \otimes E_0^\vee \rightarrow E^\vee) \in M_H(R_{v_0}(v^\vee)) \end{array} \right. \right\}. \quad (3.29)$$

Then, by using Lemma 1.5, Lemma 3.1 and (3.28), we see that $\operatorname{codim}_{M_H(v)}(M_H(v) \setminus M_H(v)^*) \geq 2$. Since Theorem 0.1 holds for $M_H(R_{v_0}(v^\vee))$, $M_H(R_{v_0}(v^\vee))$ is irreducible. Therefore the morphism $M_H(v)^* \rightarrow M_H(R_{v_0}(v^\vee))$ is birational, if $M_H(v) \neq \emptyset$. Conversely, for a μ -stable vector bundle $F \in M_H(R_{v_0}(v^\vee))$, we consider the universal extension

$$0 \rightarrow \operatorname{Ext}^1(F, E_0^\vee)^\vee \otimes E_0^\vee \rightarrow E' \rightarrow F \rightarrow 0. \quad (3.30)$$

We claim that $(E')^\vee$ is a stable sheaf of $v((E')^\vee) = v$. This means that $M_H(v) \neq \emptyset$ and Theorem 0.1 (1), (2-1) hold for this case. The proof of Theorem 0.1 (2-2) is similar to the proof for case A.

Proof of the claim: Assume that $(E')^\vee$ is not stable. Since $(E')^\vee$ is a μ -semi-stable vector bundle, there is a subbundle G such that

- (1) $(E')^\vee/G$ is torsion free,
- (2) $v(G) = l_1 v_0 - a_1 \omega$ and
- (3) $a_1/l_1 < a/l$.

Since F^\vee is μ -stable and $(E')^\vee/G$ is torsion free, $F^\vee \cap G = 0$, or F^\vee . If $F^\vee \cap G = 0$, then $G \rightarrow \operatorname{Ext}^1(F, E_0) \otimes E_0$ is injective. Since the slopes of G and E_0 are the same and G is locally free, we see that $G \cong E_0^{\oplus l_1}$, which means that (3.30) is not the universal extension. Therefore $F^\vee \cap G$ must be equal to F^\vee . This means that G contains F^\vee and $v(G/F^\vee) = (l_1 + l - a \operatorname{rk} v_0)v_0 - (a_1 - a)\omega$. Since G/F^\vee is a subsheaf of $\operatorname{Ext}^1(F, E_0) \otimes E_0$, $a_1 - a \geq 0$. On the other hand, (3) and $l_1 \leq l$ implies that $a_1 < a$, which is a contradiction. Hence $(E')^\vee$ is stable. \square

3.1.3. *Non-locally free components.* We shall prove Theorem 0.1 for

$$v = \begin{cases} (\text{rk } v_0)v_0 - \omega, \\ lv_0 - (l+1)\omega, \text{ rk } v_0 = 1. \end{cases} \quad (3.31)$$

Proposition 3.2. *Assume that $v = (\text{rk } v_0)v_0 - \omega$. Then $M_H(v) \cong X$.*

Proof. By the argument in (I-b-2), $E \in M_H(v)$ satisfies that $E^{\vee\vee} = E_0^{\oplus \text{rk } v_0}$. Since $v = v(E_0^{\oplus \text{rk } v_0}) - \omega$, E is the kernel of a quotient $E_0^{\oplus \text{rk } v_0} \rightarrow \mathbb{C}_x, x \in X$. We shall construct a family of stable sheaves $\{\mathcal{E}_x\}_{x \in X}$ of $v(\mathcal{E}_x) = v$ and prove our proposition. For convenience sake, we set $X_i := X, i = 1, 2$. Let $\Delta \subset X_2 \times X_1$ be the diagonal. We denote the projections $X_2 \times X_1 \rightarrow X_i, i = 1, 2$ by p_i . We shall consider the evaluation map

$$\phi : E_0^\vee \boxtimes E_0 \rightarrow (E_0^\vee \boxtimes E_0)|_\Delta \rightarrow \mathcal{O}_\Delta. \quad (3.32)$$

Then it is easy to see that $\phi_x := \phi|_{\{x\} \times X_1}, x \in X_2$ is surjective and the induced homomorphism

$$\text{Hom}(E_0, (E_0^\vee \boxtimes E_0)|_{\{x\} \times X}) \rightarrow \text{Hom}(E_0, \mathbb{C}_x) \quad (3.33)$$

is an isomorphism. Hence $\ker \phi_x$ is stable. Since \mathcal{O}_Δ is flat over X_2 , $\mathcal{E} := \ker \phi$ is flat over X_2 and $\mathcal{E}|_{\{x\} \times X} = \ker \phi_x, x \in X_2$ is stable. Thus we get a morphism $X_2 \rightarrow M_H(v)$, which is an isomorphism. \square

Corollary 3.3. *R_{v_0} is the Fourier-Mukai transform defined by \mathcal{E} .*

Proof. Let $\mathcal{F} : \mathbf{D}(X_1) \rightarrow \mathbf{D}(X_2)$ be the functor defined by

$$\mathcal{F}(x) := \mathbf{R} \text{Hom}_{p_2}(\mathcal{E}, p_1^*(x)), x \in \mathbf{D}(X_1) \quad (3.34)$$

and $\widehat{\mathcal{F}} : \mathbf{D}(X_2) \rightarrow \mathbf{D}(X_1)$ the functor defined by

$$\widehat{\mathcal{F}}(y) := \mathbf{R}p_{1*}(\mathcal{E} \otimes p_2^*(y)), y \in \mathbf{D}(X_2). \quad (3.35)$$

Then $\widehat{\mathcal{F}}[2]$ gives the inverse of \mathcal{F} . Let E be a coherent sheaf on X_1 such that

$$\begin{aligned} \text{Hom}(E_0, E) &= 0, \\ \text{Ext}^2(E_0, E) &= 0. \end{aligned} \quad (3.36)$$

We shall prove that E satisfies WIT_1 for \mathcal{F} , i.e.

$$\text{Hom}_{p_2}(\mathcal{E}, p_1^*(E)) = \text{Ext}_{p_2}^2(\mathcal{E}, p_1^*(E)) = 0 \quad (3.37)$$

and $\mathcal{F}^1(E) := \text{Ext}_{p_2}^1(\mathcal{E}, p_1^*(E))$ fits in the universal extension of E by E_0 : Since \mathcal{O}_Δ is flat over X_1 and K_X is trivial, we get

$$\mathcal{H}om^i(\mathcal{O}_\Delta, p_1^*(E)) = \begin{cases} 0, & i = 0, 1, \\ \mathcal{O}_\Delta \otimes p_1^*(E), & i = 2. \end{cases} \quad (3.38)$$

By using local-global spectral sequence, (3.36) and (3.38), we see that (3.37) holds and we get an exact sequence

$$0 \rightarrow \text{Ext}^1(E_0, E) \otimes E_0 \rightarrow \text{Ext}_{p_2}^1(\mathcal{E}, p_1^*(E)) \rightarrow E \rightarrow 0. \quad (3.39)$$

We claim that this sequence gives the universal extension of E by E_0 .

Proof of the claim: If it is not the universal extension, since $\text{Ext}^1(E_0, E) \cong \text{Ext}^1(E, E_0)^\vee$, E_0 must be a direct summand of $\mathcal{F}^1(E)$. Since E satisfies WIT_1 for \mathcal{F} , $\mathcal{F}^1(E)$ satisfies WIT_1 for $\widehat{\mathcal{F}}$, which implies that E_0 also satisfies WIT_1 for $\widehat{\mathcal{F}}$. In particular, $R^2p_{1*}(\mathcal{E} \otimes p_2^*(E_0)) = 0$. On the other hand, a direct computation shows that $R^2p_{1*}(\mathcal{E} \otimes p_2^*(E_0)) \cong E_0$, which is a contradiction. Therefore (3.39) is the universal extension. Thus $\mathcal{F}^1(E)$ is the reflection of E by v_0 . \square

Remark 3.1. As in [Y7], we define $\mathcal{F}^H : H^*(X_1, \mathbb{Z}) \rightarrow H^*(X_2, \mathbb{Z})$ by

$$\mathcal{F}^H(x) := p_{2*} \left(\text{ch}(\mathcal{E})^\vee p_1^* \sqrt{\text{td}_{X_1}} p_2^* \sqrt{\text{td}_{X_2}} p_1^*(x) \right), x \in H^*(X_1, \mathbb{Z}). \quad (3.40)$$

Since $\text{ch}(\mathcal{E})^\vee = p_1^*(\text{ch}(E_0))^\vee p_2^*(\text{ch}(E_0)) - \text{ch}(\mathcal{O}_\Delta)^\vee$, we also see that $\mathcal{F}^H(x) = -(x + \langle x, v(E_0) \rangle v(E_0)) = -R_{v(E_0)}(x)$. Thus we get the following commutative diagram:

$$\begin{array}{ccc} \mathbf{D}(X_1) & \xrightarrow{\mathcal{F}} & \mathbf{D}(X_2) \\ \text{ch} \sqrt{\text{td}_{X_1}} \downarrow & & \downarrow \text{ch} \sqrt{\text{td}_{X_2}} \\ H^*(X_1, \mathbb{Z}) & \xrightarrow{-R_{v(E_0)}} & H^*(X_2, \mathbb{Z}) \end{array} \quad (3.41)$$

Finally we shall treat $M_H(lv_0 - (l+1)\omega)$, $\text{rk } v_0 = 1$.

Proposition 3.4. *If $\text{rk } v_0 = 1$, then $M_H(lv_0 - (l+1)\omega) \cong \text{Hilb}_X^{l+1}$ and θ_v is an isometry of Hodge structures.*

Proof. We may assume that $v_0 = 1 + \omega$. Let E be an element of $M_H(l - \omega)$. We shall first prove that $E^{\vee\vee} \cong \mathcal{O}_X^{\oplus l}$. To see this, it is sufficient to prove that

$$n := \dim(E^{\vee\vee}/E) = l + 1. \quad (3.42)$$

Proof of (3.42): Since $\chi(E) = l - 1$ and E is stable, Serre duality implies that $\dim H^0(X, E^\vee) \geq l - 1$. Hence we get an exact sequence

$$0 \rightarrow \mathcal{O}_X^{\oplus(l-1)} \rightarrow E^\vee \rightarrow I_Z \rightarrow 0 \quad (3.43)$$

where $I_Z \in \text{Hilb}_X^{l+1-n}$. If $n = 0$, then E is locally free. By taking the dual of (3.43), we get a section of E , which contradicts the stability of E . We assume that $0 < n < l + 1$. Then $\dim \text{Ext}^1(I_Z, \mathcal{O}_X) = \dim H^1(X, I_Z) = l - n$. Hence we get a decomposition $E^\vee = \mathcal{O}_X^{\oplus(n-1)} \oplus F$, where F fits in an exact sequence

$$0 \rightarrow \mathcal{O}_X^{\oplus(l-n)} \rightarrow F \rightarrow I_Z \rightarrow 0. \quad (3.44)$$

Then $E^{\vee\vee}$ has a subsheaf $\mathcal{O}_X^{\oplus n}$. The stability of E implies that

$$\frac{\chi(\mathcal{O}_X^{\oplus n} \cap E)}{n} < \frac{\chi(E)}{l}. \quad (3.45)$$

Since $\chi(\mathcal{O}_X^{\oplus n} \cap E) \geq \chi(\mathcal{O}_X^{\oplus n}) - n = n$, this is impossible. Therefore (3.43) holds, which implies that $E^{\vee\vee} = \mathcal{O}_X^{\oplus l}$.

Conversely for a general quotient $\phi : \mathcal{O}_X^{\oplus l} \rightarrow \bigoplus_{i=1}^{l+1} \mathbb{C}_{x_i}$, $x_1, x_2, \dots, x_{l+1} \in X$, it is easy to see that $\ker \phi$ is stable. Thus $M_H(l - \omega) \neq \emptyset$.

We shall prove that it is isomorphic to Hilb_X^{l+1} . For this purpose, we shall consider a functor $\mathcal{G} : \mathbf{D}(X_1) \rightarrow \mathbf{D}(X_2)_{op}$ which is the composition of reflection by $v(\mathcal{O}_X)$ with the taking dual functor:

$$\mathcal{G}(x) := \mathbf{R} \text{Hom}_{p_2}(p_1^*(x), I_\Delta), x \in \mathbf{D}(X_1), \quad (3.46)$$

where we use the same notation as in Proposition 3.2 and $\mathbf{D}(X_2)_{op}$ is the opposite category of $\mathbf{D}(X_2)$. Then \mathcal{G} gives an equivalence of categories. For $E \in M_H(l - \omega)$, we shall prove that

- (a) $\text{Ext}_{p_2}^i(p_1^*(E), I_\Delta) = 0$, $i = 0, 2$ and
- (b) $\mathcal{G}^1(E) := \text{Ext}_{p_2}^1(p_1^*(E), I_\Delta)$ is an ideal sheaf of colength $l + 1$.

Then the map $M_H(l - \omega) \rightarrow \text{Hilb}_X^{l+1}$ sending E to $\mathcal{G}^1(E)$ gives an isomorphism of moduli spaces. The second assertion follow from [Y7, Prop. 2.5] or a direct computation by using the equality

$$\mathcal{G}(\mathcal{E}) = \mathbf{R} \text{Hom}_{p_{M_H(l-\omega)}}(\mathcal{E}, \mathcal{O}_{M_H(l-\omega) \times X_1}) \boxtimes \mathcal{O}_{X_2} - \mathbf{R} \mathcal{H}om(\mathcal{E}, \mathcal{O}_{M_H(l-\omega) \times X_2}) \quad (3.47)$$

as an element of Grothendieck group of $M_H(l - \omega) \times X_2$, where \mathcal{E} is a quasi-universal family on $M_H(l - \omega) \times X$.

Proof of (a), (b): By Serre duality and the stability of E , $\text{Ext}^2(E, I_x) = 0$ for all $x \in X$. Also we see that $\text{Hom}(E, I_x) = 0$ for $x \notin \text{Supp}(E^{\vee\vee}/E)$. Hence by the base change theorem and its proof, we see that (a) holds and $\mathcal{G}^1(E)$ is torsion free. It is easy to see that $v(\mathcal{G}(E)) = -R_{v(\mathcal{O}_X)}(v(E)^\vee) = -1 + l\omega$. Hence $v(\mathcal{G}^1(E)) = 1 - l\omega$. Therefore $\mathcal{G}^1(E)$ is an ideal sheaf of colength $l + 1$. \square

3.2. The case of $\langle v^2 \rangle = 2l^2$. We next treat the case where $\langle v^2 \rangle = 2l^2$. Let E be a general member of $M_H(v) \setminus M_H(v)^{\mu s}$. Then by the proof of Lemma 1.5, E fits in an exact sequence

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_0 \rightarrow 0 \quad (3.48)$$

where E_1 is a μ -stable vector bundle.

Indeed, if $\langle v^2 \rangle = 2l^2$, then the primitivity of v implies that (i) $v = lv_0 - 2\omega$, $l = \text{rk } v_0$ or (ii) $v = lv_0 - \omega$, $2l = \text{rk } v_0$. In particular $\text{rk } v_0 > 1$. In the notation of the proof of Lemma 1.5, we see that $k = 1$. Hence E fits in the above exact sequence. Then in the same way as in 2.2.2, we get Theorem 0.1 in this case. \square

3.3. Some remarks. By the proof of Theorem 0.1 for case B and Lemma 1.5, we also get the following.

- (1) $M_H(v)^{\mu s} \neq \emptyset$ if and only if $\langle v^2 \rangle \geq 2l^2$.
- (2) $M_H(v)^{loc} = \emptyset$ if and only if (i) $\text{rk } v = 1$, (ii) $v = (\text{rk } v_0)v_0 - \omega$, or (iii) $\text{rk } v_0 = 1$ and $v = lv_0 - (l+1)\omega$.

Combining Propositions 3.4, 3.2 and Remark 2.2, we get Proposition 0.5.

4. RELATION TO MONTONEN-OLIVE DUALITY

In this section, we shall consider the relation between Theorem 0.1 and Montonen-Olive duality in Physics. Roughly speaking, Montonen-Olive duality says that the generating function of Euler characteristics of moduli spaces of vector bundles becomes a modular form. In this paper, we concentrate on moduli spaces of vector bundles on K3 surfaces. We first describe physical predictions and their modifications. For more details and related results, see [MNVW], [V-W] and [Gö3], [Y4], [Y6].

4.1. Physical predictions. We fix a K3 surface X . We regard $H^2(X, \mathbb{Z})$ as a lattice by a bilinear form $Q(x, y) = -\int_X xy, x, y \in H^2(X, \mathbb{Z})$. Let P be a orthogonal decomposition of $H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ as a sum of definite signature:

$$P : H^2(X, \mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}^{19,0} \oplus \mathbb{R}^{0,3}. \quad (4.1)$$

Let $P_L(x) = x_L, P_R(x) = x_R$ denote the projections onto the two factors.

For $v = r + \xi + a\omega \in H^*(X, \mathbb{Z})$ of $\xi \in H^2(X, \mathbb{Z})$, we choose a suitable complex structure such that ξ become holomorphic. Then we define $M(v)$ as a moduli space of stable sheaves on this surface. Let $Z_r(\tau, x)$ be $U(r)$ -partition function defined in [MNVW, sect. 3]:

$$Z_r(\tau, x) := \sum_{\substack{v \in H^*(X, \mathbb{Z}) \\ \text{rk } v = r}} \text{“}\chi(M(v))\text{”} q^{\frac{\langle v^2 \rangle}{2r}} q^{\frac{1}{2r} Q(c_1(v)_L^2)} \bar{q}^{\frac{-1}{2r} Q(c_1(v)_R^2)} e^{Q(c_1(v), x)} \quad (4.2)$$

where $(\tau, x) \in \mathbb{H} \times H^2(X, \mathbb{Z}) \otimes \mathbb{C}$, $q := \exp(2\pi\sqrt{-1}\tau)$, $e := \exp(2\pi\sqrt{-1})$ and “ $\chi(M(v))$ ” is a kind of “Euler characteristics” of a nice compactification of $M(v)$.

Remark 4.1. More precisely, Minahan et al. considered $Z_r(\tau, 0)$. Combining the computations in [MNVW, sect. 6], we propose the definition (4.2).

Unfortunately, there is no mathematical definition of this “Euler characteristics”. Since $M(v)$ is smooth and compact for primitive v , we can expect that “ $\chi(M(v))$ ” coincides with the ordinary Euler characteristics $\chi(M(v))$.

Then Montonen-Olive duality for $U(r)$ gauge group asserts that

(#) $Z_r(\tau, x)$ transforms like a Jacobi form of holomorphic/anti-holomorphic weight

$$(-\chi(X)/2 + b_-(X)/2, b_+(X)/2) = (-5/2, 3/2)$$

(cf. [E-Z]). For $\alpha \in H^2(X, \mathbb{Z})$, let $Z_r^\alpha(\tau)$ be $PSU(r)$ -partition function defined in [V-W]:

$$Z_r^\alpha(\tau) := \sum_{\substack{v \in H^*(X, \mathbb{Z}) \\ \text{rk } v = r, c_1(v) = \alpha}} \text{“}\chi(M(v))\text{”} q^{\frac{\langle v^2 \rangle}{2r}}. \quad (4.3)$$

Then

$$Z_r(\tau, x) = \sum_{\alpha \in H^2(X, \mathbb{Z})/rH^2(X, \mathbb{Z})} Z_r^\alpha(\tau) \Theta_{\alpha, r}(\tau, P, x), \quad (4.4)$$

where

$$\Theta_{\alpha, r}(\tau, P, x) = \sum_{c \in \alpha + rH^2(X, \mathbb{Z})} q^{\frac{1}{2r} Q(c_L^2)} \bar{q}^{\frac{-1}{2r} Q(c_R^2)} e^{Q(c, x)} \quad (4.5)$$

is Siegel-Narain theta function (cf. [M-W, Appendix B]). If $r = 1$, then it is known that $Z_1^0(\tau) = \frac{1}{\eta(\tau)^{24}}$ ([Gö1], [V-W]). Hence

$$\begin{aligned} Z_1(\tau, x) &= Z_1^0(\tau) \left(\sum_{c \in H^2(X, \mathbb{Z})} q^{\frac{1}{2} Q(c_L^2)} \bar{q}^{\frac{-1}{2} Q(c_R^2)} e^{Q(c, x)} \right) \\ &= \frac{1}{\eta(\tau)^{24}} \Theta(\tau, P, x), \end{aligned} \quad (4.6)$$

where $\Theta(\tau, P, x) = \sum_{c \in H^2(X, \mathbb{Z})} q^{\frac{1}{2} Q(c_L^2)} \bar{q}^{\frac{-1}{2} Q(c_R^2)} e^{Q(c, x)}$. Since $\Theta(\tau, P, x)$ transforms like a Jacobi form of holomorphic/anti-holomorphic weight $(19/2, 3/2)$, $Z_1(\tau, x)$ transforms like a Jacobi form of holomorphic/anti-holomorphic weight $(-5/2, 3/2)$:

$$Z_1\left(-\frac{1}{\tau}, \frac{x_L}{\tau} + \frac{x_R}{\bar{\tau}}\right) = (-\sqrt{-1}\tau)^{-5/2} (\sqrt{-1}\bar{\tau})^{3/2} e^{\frac{Q(x_L^2)}{2\tau}} e^{\frac{Q(x_R^2)}{2\bar{\tau}}} Z_1(\tau, x_L + x_R). \quad (4.7)$$

Then $Z_r(\tau, x)$ is given by Hecke transformation of order r of $Z_1(\tau, x)$ ([MNVW]):

$$Z_r(\tau, x) = \frac{1}{r^2} \sum_{\substack{a, b, d \geq 0 \\ ad=r \\ b < d}} dZ_1\left(\frac{a\tau + b}{d}, ax\right). \quad (4.8)$$

In particular, $Z_r(\tau, x)$ transforms like a Jacobi form of holomorphic/anti-holomorphic weight $(-5/2, 3/2)$ and index r . Thus (#) holds.

Remark 4.2. For $PSU(r)$ -partition functions, we get the following:

$$Z_r^{c_1}(\tau) = \frac{1}{r^2} \sum_{\substack{a, b, d \geq 0 \\ ad=r \\ b < d \\ a\xi=c_1}} dZ_1^0\left(\frac{a\tau + b}{d}\right) e^{(-\frac{b}{2d}(\xi^2))}. \quad (4.9)$$

Combining (4.4) with the transformation law

$$\Theta_{\alpha, r}\left(\frac{-1}{\tau}, P, \frac{x_L}{\tau} + \frac{x_R}{\bar{\tau}}\right) = r^{-11}(-\sqrt{-1}\tau)^{19/2}(\sqrt{-1}\bar{\tau})^{3/2} e^{\frac{rQ(x_L^2)}{2\tau}} e^{\frac{rQ(x_R^2)}{2\bar{\tau}}} \cdot \left(\sum_{\beta \in H^2(X, \mathbb{Z}/r\mathbb{Z})} e^{\frac{-Q(\alpha, \beta)}{r}} \Theta_{\beta, r}(\tau, x_L + x_R) \right), \quad (4.10)$$

we can deduce from (#) the following transformation law:

$$Z_r^\alpha(-1/\tau) = r^{-11}(-\sqrt{-1}\tau)^{-12} \sum_{\beta \in H^2(X, \mathbb{Z}/r\mathbb{Z})} e^{\frac{Q(\alpha, \beta)}{r}} Z_r^\beta(\tau). \quad (4.11)$$

This formula is of course compatible with Montonen-Olive duality for $PSU(r)$ group [V-W].

4.2. Relation to Theorem 0.1. We shall check that Theorem 0.1 is compatible with (4.8). For simplicity, we set $X^{[n]} = \text{Hilb}_X^n$.

$$\begin{aligned} \sum_{0 \leq b < d} dZ_1\left(\frac{a\tau + b}{d}, ax\right) &= \sum_{0 \leq b < d} \sum_{\xi \in H^2(X, \mathbb{Z})} \sum_n d\chi(X^{[n]}) q^{\frac{a}{d}(n-1)} q^{\frac{a}{2d}Q(\xi_L^2)} \bar{q}^{\frac{-a}{2d}Q(\xi_R^2)} e^{aQ(\xi, x)} e^{\frac{b}{d}((n-1)+Q(\xi^2)/2)} \\ &= \sum_{\xi \in H^2(X, \mathbb{Z})} \sum_{d|n-1+\frac{Q(\xi^2)}{2}} d^2\chi(X^{[n]}) q^{\frac{a}{d}(n-1)} q^{\frac{a}{2d}Q(\xi_L^2)} \bar{q}^{\frac{-a}{2d}Q(\xi_R^2)} e^{aQ(\xi, x)} \\ &= \sum_{\xi \in H^2(X, \mathbb{Z})} \sum_k d^2\chi(X^{[kd-Q(\xi^2)/2+1]}) q^{\frac{a}{d}(kd-Q(\xi^2)/2)} q^{\frac{a}{2d}Q(\xi_L^2)} \bar{q}^{\frac{-a}{2d}Q(\xi_R^2)} e^{aQ(\xi, x)} \\ &= \sum_{\xi \in H^2(X, \mathbb{Z})} \sum_{w=(d, \xi, -k)} d^2\chi(X^{[\langle w^2 \rangle/2+1]}) q^{\frac{a}{2d}\langle w^2 \rangle} q^{\frac{a}{2d}Q(c_1(w)_L^2)} \bar{q}^{\frac{-a}{2d}Q(c_1(w)_R^2)} e^{aQ(c_1(w), x)} \\ &= \sum_{\text{rk } w=d} d^2\chi(X^{[\langle w^2 \rangle/2+1]}) q^{\frac{1}{2r}\langle (aw)^2 \rangle} q^{\frac{1}{2r}Q(c_1(aw)_L^2)} \bar{q}^{\frac{-1}{2r}Q(c_1(aw)_R^2)} e^{Q(c_1(aw), x)}. \end{aligned} \quad (4.12)$$

Therefore we get

$$“\chi(M(v))” = \sum_{v=aw} \frac{1}{a^2} \chi(X^{[\langle w^2 \rangle/2+1]}). \quad (4.13)$$

In particular, if v is primitive, then by Corollary 0.2, we get

$$“\chi(M(v))” = \chi(X^{[\langle v^2 \rangle/2+1]}) = \chi(M(v)). \quad (4.14)$$

This implies that $\chi(M(v))$ is related to modular forms and in particular Hecke transforms. To understand the meaning of “ $\chi(M(v))$ ” for non-primitive v is a challenging problem. The relation to O’Grady’s symplectic compactification of $M(2-2\omega)$ ([O2]) is also an interesting problem.

5. APPENDIX

In this appendix, we shall explain our method for dimension counting of substacks of $\mathcal{M}(v)^{\mu_{ss}}$. Since most results are appeared in another forms (cf. [D-R], [H-N]), we only give an outline.

5.1. Notation. For an ample divisor H' on X , let $Q(mH', v)$ be the open subscheme of the quot-scheme $\text{Quot}_{\mathcal{O}_X(-mH')^{\oplus N}/X/\mathbb{C}}$ consisting of points

$$\lambda : \mathcal{O}_X(-mH')^{\oplus N} \rightarrow E \quad (5.1)$$

such that

1. $v(E) = v$,
2. λ induces an isomorphism $H^0(X, \mathcal{O}_X^{\oplus N}) \cong H^0(X, E(mH'))$,
3. $H^i(X, E(mH')) = 0$, $i > 0$.

Let $\mathcal{Q}_{Q(mH', v) \times X}(-mH')^{\oplus N} \rightarrow \mathcal{Q}_v$ be the universal quotient. We set $V_v := \mathcal{O}_X(-mH')^{\oplus N}$. For our purpose, the choice of mH is not so important. Hence we simply denote $Q(mH', v)$ by $Q(v)$.

Let $q_v : Q(v) \rightarrow \mathcal{M}(v)$ be the natural map. We denote the pull-backs $q_v^{-1}(\mathcal{M}(v)^{\mu ss})$, $q_v^{-1}(\mathcal{M}(v)^{ss})$, \dots by $Q(v)^{\mu ss}$, $Q(v)^{ss}$, \dots respectively. If we choose a suitable $Q(v)$, then $q_v : Q(v)^{\mu ss} \rightarrow \mathcal{M}(v)^{\mu ss}$ is surjective and $\mathcal{M}(v)^{\mu ss}$ is a quotient stack of $Q(v)^{\mu ss}$ by a natural action of $G_v := GL(N)$:

$$\mathcal{M}(v)^{\mu ss} \cong [Q(v)^{\mu ss}/G_v]. \quad (5.2)$$

From now on, we assume that $q_v : Q(v)^{\mu ss} \rightarrow \mathcal{M}(v)^{\mu ss}$ is surjective.

5.2. Stack of filtrations.

Definition 5.1. $\mathcal{F}(v_1, v_2)$ is the stack of filtrations $F_1 \subset E$, $E \in \mathcal{M}(v)$ such that

1. F_1 is a μ -semi-stable sheaf of $v(F_1) = v_1$.
2. E/F_1 is a μ -semi-stable sheaf of $v(E/F_1) = v_2$.

Let $p_v : \mathcal{F}(v_1, v_2) \rightarrow \mathcal{M}(v)^{\mu ss}$ be the projection sending $(F_1 \subset E)$ to E and $p_{v_1, v_2} : \mathcal{F}(v_1, v_2) \rightarrow \mathcal{M}(v_1)^{\mu ss} \times \mathcal{M}(v_2)^{\mu ss}$ the morphism sending $(F_1 \subset E)$ to $(F_1, E/F_1)$.

We consider an open subscheme $F(v_1, v_2)$ of $\text{Quot}_{\mathcal{Q}_v/Q(v)^{\mu ss} \times X/Q(v)^{\mu ss}}$ consisting of quotients $(\mathcal{Q}_v)_x \rightarrow E_2$, $x \in Q(v)^{\mu ss}$ such that E_2 is a μ -semi-stable sheaf of $v(E_2) = v_2$. Then

$$\mathcal{F}(v_1, v_2) = [F(v_1, v_2)/G_v]. \quad (5.3)$$

We shall give another expression of $\mathcal{F}(v_1, v_2)$ which is useful to compute the dimensions of substacks of $\mathcal{F}(v_1, v_2)$ and its projections to $\mathcal{M}(v)$.

We shall choose $Q(mH', v)^{\mu ss}$, $Q(mH', v_1)^{\mu ss}$ and $Q(mH', v_2)^{\mu ss}$ for the same mH' . Then $V_v = V_{v_1} \oplus V_{v_2}$. For simplicity, we set $\mathcal{Q}_i := \mathcal{Q}_{v_i}$, $V_i := V_{v_i}$, \dots and \mathcal{K}_i are the universal subsheaves of $\mathcal{O}_{Q(v_i)^{\mu ss}} \otimes V_i$, $i = 1, 2$. We define a scheme $\varpi : Y \rightarrow Q(v_1)^{\mu ss} \times Q(v_2)^{\mu ss}$ by

$$Y := \{\psi : (\mathcal{K}_2)_{x_2} \rightarrow (\mathcal{Q}_1)_{x_1} \mid (x_1, x_2) \in Q(v_1)^{\mu ss} \times Q(v_2)^{\mu ss}\}. \quad (5.4)$$

Then Y parameterizes subsheaves $K \subset V$ such that $K \cap V_1 = (\mathcal{K}_1)_{x_1}$ and $K/K \cap V_1 = (\mathcal{K}_2)_{x_2}$: For a quotient $\psi : (\mathcal{K}_2)_{x_2} \rightarrow (\mathcal{Q}_1)_{x_1}$, the subsheaf K of V_v is defined by

$$K := \{(a_1, a_2) \in V_1 \oplus V_2 \mid a_2 \in (\mathcal{K}_2)_{x_2}, \psi(a_2) = a_1 \bmod (\mathcal{K}_1)_{x_1}\}. \quad (5.5)$$

Considering the quotient $V_v \rightarrow V_v/K$, Y also parameterizes the following exact and commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & (\mathcal{Q}_1)_{x_1} & \longrightarrow & E & \longrightarrow & (\mathcal{Q}_2)_{x_2} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & V_1 & \longrightarrow & V_v & \longrightarrow & V_2 \longrightarrow 0 \end{array} \quad (5.6)$$

Let $\xi : Y \times G_v \rightarrow F(v_1, v_2)$ be the morphism sending $(y : V_v \rightarrow E, g) \in Y \times G_v$ to

$$(y \circ g : V_v \rightarrow E, E \rightarrow \mathcal{Q}_{x_2}) \in F(v_1, v_2), \quad (5.7)$$

where $\varpi(y) = (x_1, x_2)$. Let P be the parabolic subgroup of G_v fixing V_1 . Then Y has a natural action of P and ξ induces a morphism $Y \times_P G_v \rightarrow F(v_1, v_2)$, which is G_v -equivariant. It is easy to see that this morphism is an isomorphism (cf. [Y2, appendix]). Therefore

$$\begin{aligned} \mathcal{F}(v_1, v_2) &\cong [Y \times_P G_v / G_v] \\ &\cong [Y / P]. \end{aligned} \quad (5.8)$$

By using (5.8), we shall prove the following.

Lemma 5.1. *We set*

$$\begin{aligned}\mathcal{N}^n(v_1, v_2) &:= \{(E_1, E_2) \in \mathcal{M}(v_1)^{\mu ss} \times \mathcal{M}(v_2)^{\mu ss} \mid \dim \operatorname{Hom}(E_1, E_2) = n\}, \\ \mathcal{F}^n(v_1, v_2) &:= p_{v_1, v_2}^{-1}(\mathcal{N}^n(v_1, v_2)) \\ &= \{(F_1 \subset E) \in \mathcal{F}(v_1, v_2) \mid \dim \operatorname{Hom}(F_1, E/F_1) = n\}.\end{aligned}\tag{5.9}$$

Then,

$$\dim \mathcal{F}^n(v_1, v_2) = \dim \mathcal{N}^n(v_1, v_2) + \langle v_1, v_2 \rangle + n.\tag{5.10}$$

Proof. We set

$$Q^n(v_1, v_2) := \{(x_1, x_2) \in Q(v_1)^{\mu ss} \times Q(v_2)^{\mu ss} \mid \dim \operatorname{Hom}((\mathcal{Q}_1)_{x_1}, (\mathcal{Q}_2)_{x_2}) = n\}.\tag{5.11}$$

For $(x_1, x_2) \in Q^n(v_1, v_2)$, there is an exact sequence

$$0 \rightarrow \operatorname{Hom}((\mathcal{Q}_2)_{x_2}, (\mathcal{Q}_1)_{x_1}) \rightarrow \operatorname{Hom}(V_2, (\mathcal{Q}_1)_{x_1}) \rightarrow \operatorname{Hom}((\mathcal{K}_2)_{x_2}, (\mathcal{Q}_1)_{x_1}) \rightarrow \operatorname{Ext}^1((\mathcal{Q}_2)_{x_2}, (\mathcal{Q}_1)_{x_1}) \rightarrow 0.\tag{5.12}$$

Since $\dim \operatorname{Hom}(V_2, (\mathcal{Q}_1)_{x_1}) = \operatorname{rk} V_1 \operatorname{rk} V_2$ and $\dim \operatorname{Ext}^2((\mathcal{Q}_2)_{x_2}, (\mathcal{Q}_1)_{x_1}) = n$, $\operatorname{Hom}(\mathcal{K}'_2, \mathcal{Q}'_1)$ is a locally free sheaf of rank $\langle v_1, v_2 \rangle + \operatorname{rk} V_1 \operatorname{rk} V_2 + n$ on $Q^n(v_1, v_2)$, where \mathcal{K}'_2 and \mathcal{Q}'_1 are pull-backs of \mathcal{K}_2 and \mathcal{Q}_1 to $Q^n(v_1, v_2)$ respectively. We set $Y^n := \mathbb{V}(\operatorname{Hom}(\mathcal{K}'_2, \mathcal{Q}'_1)^\vee) \rightarrow Q^n(v_1, v_2)$. Then $\mathcal{F}^n(v_1, v_2) \cong [Y^n/P]$. Hence we get that

$$\begin{aligned}\dim \mathcal{F}^n(v_1, v_2) &= \dim Y^n - \dim P \\ &= \dim Q^n(v_1, v_2) + \langle v_1, v_2 \rangle + n - (\dim G_1 + \dim G_2) \\ &= \dim \mathcal{N}^n(v_1, v_2) + \langle v_1, v_2 \rangle + n.\end{aligned}\tag{5.13}$$

□

By similar method as in the proof of Lemma 5.1, we can also prove the following.

Lemma 5.2. *Let $\mathcal{F}^0(v_1, v_2, \dots, v_s)$ be the stack of filtrations*

$$0 \subset F_1 \subset F_2 \subset \dots \subset F_s = E, E \in \mathcal{M}(v)\tag{5.14}$$

such that

1. F_i/F_{i-1} , $1 \leq i \leq s$ are semi-stable of $v(F_i/F_{i-1}) = v_i$.
2. $\operatorname{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0$, $i < j$.

Then

$$\dim \mathcal{F}^0(v_1, v_2, \dots, v_s) = \sum_{i=1}^s \mathcal{M}(v_i)^{ss} + \sum_{i < j} \langle v_i, v_j \rangle.\tag{5.15}$$

5.3. Supplement for the proof of Lemma 1.8. We shall explain how to derive (1.15) from (1.14). Let $q_l : Q(lw)^{ss} \rightarrow \overline{M}_H(lw) := Q(lw)^{ss}/G_{lw}$ be the quotient map. For a sequence of positive integers $l_1 \leq l_2 \leq \dots \leq l_s$ of $\sum_{i=1}^s l_i = l$, we set

$$\begin{aligned}\overline{M}_H(lw; l_1, l_2, \dots, l_s) &:= \{\oplus_{i=1}^s E_i^{\oplus l_i} \in \overline{M}_H(lw) \mid E_1, E_2, \dots, E_s \in M_H(w), E_i \neq E_j \text{ for } i \neq j\}, \\ Q(lw; l_1, l_2, \dots, l_s)^{ss} &:= q_l^{-1}(\overline{M}_H(lw; l_1, l_2, \dots, l_s)).\end{aligned}\tag{5.16}$$

For simplicity, we set $G := G_{lw}$ and $G_i := G_{l_i w}$, $i = 1, 2, \dots, s$. For quotients $\phi_i : V_i \rightarrow \mathcal{Q}_{x_i} \in Q(l_i w; l_i)^{ss}$, $i = 1, 2, \dots, s$ and an element $g \in G$, we define a quotient

$$(\oplus_{i=1}^s \phi_i) \circ g : V \rightarrow \oplus_{i=1}^s \mathcal{Q}_{x_i}.\tag{5.17}$$

It will define a morphism $\prod_{i=1}^s Q(l_i w; l_i)^{ss} \times G \rightarrow Q(lw)$. Let $\prod_{i=1}^s Q(l_i w; l_i)^{ss} \times_{\prod_i G_i} G$ be the quotient of $\prod_{i=1}^s Q(l_i w; l_i)^{ss} \times G$ by a natural action of $\prod_i G_i$. Then the above morphism induces a morphism

$$\pi_{l_1, l_2, \dots, l_s} : \prod_{i=1}^s Q(l_i w; l_i)^{ss} \times_{\prod_i G_i} G \rightarrow Q(lw).\tag{5.18}$$

By the construction of $\pi_{l_1, l_2, \dots, l_s}$, $\operatorname{im} \pi_{l_1, l_2, \dots, l_s}$ contains $Q(lw; l_1, l_2, \dots, l_s)^{ss}$. Hence

$$\dim [Q(lw; l_1, l_2, \dots, l_s)^{ss}/G] \leq \sum_{i=1}^s \dim [Q(l_i w; l_i)^{ss}/G_i].\tag{5.19}$$

We shall prove that

$$\dim [Q(l_i w; l_i)^{ss}/G_i] \leq 1.\tag{5.20}$$

Then $\dim [Q(lw; l_1, l_2, \dots, l_s)^{ss}/G] \leq s \leq l$. Clearly $\dim [Q(lw; 1, 1, \dots, 1)^{ss}/G] = l$. Hence $\dim [Q(lw)^{ss}/G] = l$.

Proof of (5.20): Since $q_i : Q(l_i w; l_i)^{ss} \rightarrow \overline{M}_H(l_i w; l_i)$ is surjective and $\dim \overline{M}_H(l_i w; l_i) = 2$, it is sufficient to prove that

$$\dim [q_i^{-1}(E^{\oplus l_i})/G_i] \leq -1, \quad E \in M_H(w). \quad (5.21)$$

By definition, $\mathcal{J}(l_i, E) = [q_i^{-1}(E^{\oplus l_i})/G_i]$. Hence we obtain this claim from (1.14).

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